Chapter 1: Introduction: Waves and Phasors

Lesson #1

Chapter — Section: Chapter 1

Topics: EM history and how it relates to other fields

Highlights:

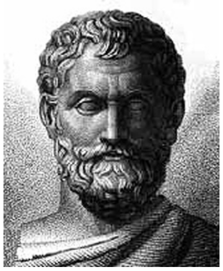
- EM in Classical era: 1000 BC to 1900
- Examples of Modern Era Technology timelines
- Concept of "fields" (gravitational, electric, magnetic)
- Static vs. dynamic fields
- The EM Spectrum

Special Illustrations:

- Timelines from CD-ROM

Timeline for Electromagnetics in the Classical Era

ca. 900 BC Legend has it that while walking across a field in northern Greece, a shepherd named Magnus experiences a pull on the iron nails in his sandals by the black rock he was standing on. The region was later named Magnesia and the rock became known as magnetite [a form of iron with permanent magnetism].



ca. 600 BC Greek philosopher **Thales** describes how amber, after being rubbed with cat fur, can pick up feathers [static electricity].

ca. 1000 Magnetic compass used as a navigational device.

1752 **Benjamin Franklin** (American) invents the lightning rod and demonstrates that lightning is electricity.

1785 **Charles-Augustin de Coulomb** (French) demonstrates that the electrical force between charges is proportional to the inverse of the square of the distance between them.

1800 **Alessandro Volta** (Italian) develops the first electric battery.

1820 **Hans Christian Oersted** (Danish) demonstrates the interconnection between electricity and magnetism through his discovery that an electric current in a wire causes a compass needle to orient itself perpendicular to the wire.

Lessons #2 and 3

Chapter — Sections: 1-1 to 1-6

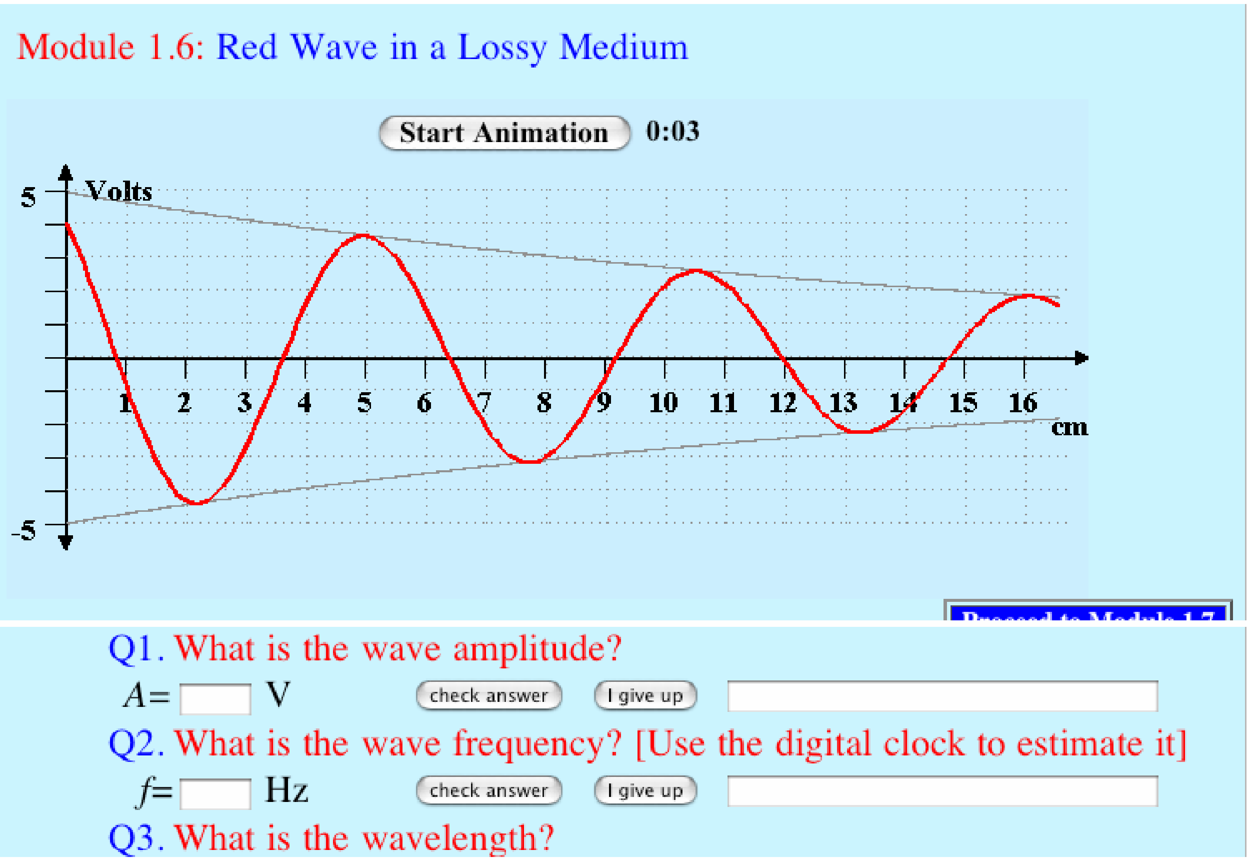
Topics: Waves

Highlights:

- Wave properties
- Complex numbers
- Phasors

Special Illustrations:

- CD-ROM Modules 1.1-1.9
- CD-ROM Demos 1.1-1.3



Module 1.6: Red Wave in a Lossy Medium

Q1. What is the wave amplitude?

$A=$ ☐ V

Q2. What is the wave frequency? [Use the digital clock to estimate it]

$f=$ ☐ Hz

Q3. What is the wavelength?

Chapter 1

Section 1-3: Traveling Waves

Problem 1.1 A 2-kHz sound wave traveling in the x-direction in air was observed to have a differential pressure $p(x,t) = 10$ N/m^2 at $x = 0$ and $t = 50$ μs. If the reference phase of $p(x,t)$ is 36°, find a complete expression for $p(x,t)$. The velocity of sound in air is 330 m/s.

Solution: The general form is given by Eq. (1.17),

$$p(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0\right),$$

where it is given that $\phi_0 = 36°$. From Eq. (1.26), $T = 1/f = 1/(2\times 10^3) = 0.5$ ms. From Eq. (1.27),

$$\lambda = \frac{u_{\text{p}}}{f} = \frac{330}{2\times 10^3} = 0.165 \text{ m.}$$

Also, since

$$\begin{aligned} p(x=0,\ t=50\ \mu\text{s}) = 10\ (\text{N/m}^2) &= A\cos\left(\frac{2\pi\times 50\times 10^{-6}}{5\times 10^{-4}} + 36°\frac{\pi \text{ rad}}{180°}\right) \\ &= A\cos(1.26 \text{ rad}) = 0.31A, \end{aligned}$$

it follows that $A = 10/0.31 = 32.36$ N/m^2. So, with t in (s) and x in (m),

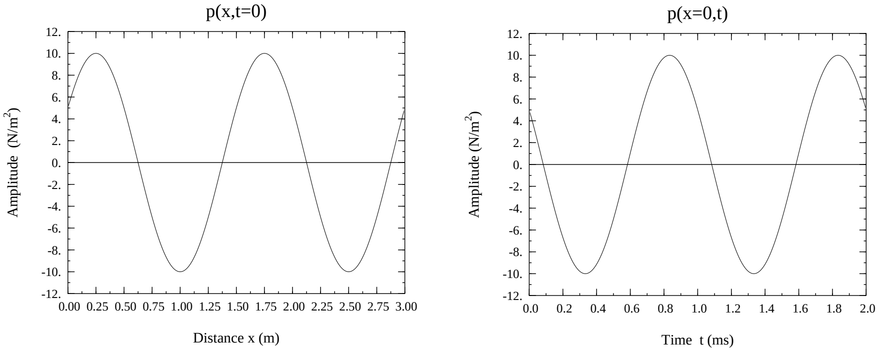
$$\begin{aligned} p(x,t) &= 32.36\cos\left(2\pi\times 10^6\frac{t}{500} - 2\pi\times 10^3\frac{x}{165} + 36°\right) \quad (\text{N/m}^2) \\ &= 32.36\cos(4\pi\times 10^3 t - 12.12\pi x + 36°) \quad (\text{N/m}^2). \end{aligned}$$

Problem 1.2 For the pressure wave described in Example 1-1, plot

- **(a)** $p(x,t)$ versus x at $t = 0$,
- **(b)** $p(x,t)$ versus t at $x = 0$.

Be sure to use appropriate scales for x and t so that each of your plots covers at least two cycles.

Solution: Refer to Fig. P1.2(a) and Fig. P1.2(b).



(a) (b)

Figure P1.2: (a) Pressure wave as a function of distance at $t = 0$ and (b) pressure wave as a function of time at $x = 0$.

Problem 1.3 A harmonic wave traveling along a string is generated by an oscillator that completes 180 vibrations per minute. If it is observed that a given crest, or maximum, travels 300 cm in 10 s, what is the wavelength?

Solution:

$$f = \frac{180}{60} = 3 \text{ Hz}.$$
$$u_\text{p} = \frac{300 \text{ cm}}{10 \text{ s}} = 0.3 \text{ m/s}.$$
$$\lambda = \frac{u_\text{p}}{f} = \frac{0.3}{3} = 0.1 \text{ m} = 10 \text{ cm}.$$

Problem 1.4 Two waves, $y_1(t)$ and $y_2(t)$, have identical amplitudes and oscillate at the same frequency, but $y_2(t)$ leads $y_1(t)$ by a phase angle of 60°. If

$$y_1(t) = 4\cos(2\pi \times 10^3 t),$$

write down the expression appropriate for $y_2(t)$ and plot both functions over the time span from 0 to 2 ms.

Solution:

$$y_2(t) = 4\cos(2\pi \times 10^3 t + 60°).$$

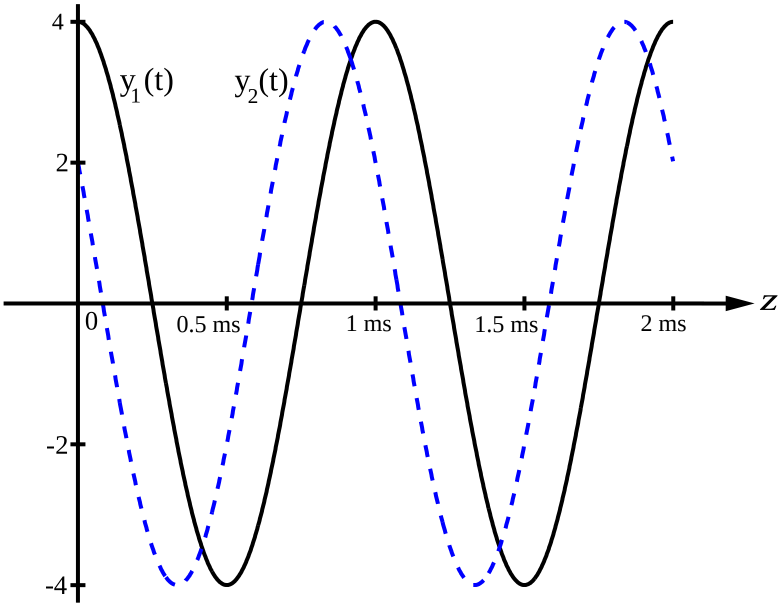


Figure P1.4: Plots of $y_1(t)$ and $y_2(t)$.

Problem 1.5 The height of an ocean wave is described by the function

$$y(x,t) = 1.5\sin(0.5t - 0.6x) \quad \text{(m)}.$$

Determine the phase velocity and the wavelength and then sketch $y(x,t)$ at $t = 2$ s over the range from $x = 0$ to $x = 2\lambda$.

Solution: The given wave may be rewritten as a cosine function:

$$y(x,t) = 1.5\cos(0.5t - 0.6x - \pi/2).$$

By comparison of this wave with Eq. (1.32),

$$y(x,t) = A\cos(\omega t - \beta x + \phi_0),$$

we deduce that

$$\omega = 2\pi f = 0.5 \text{ rad/s}, \qquad \beta = \frac{2\pi}{\lambda} = 0.6 \text{ rad/m},$$

$$u_p = \frac{\omega}{\beta} = \frac{0.5}{0.6} = 0.83 \text{ m/s}, \qquad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.6} = 10.47 \text{ m}.$$

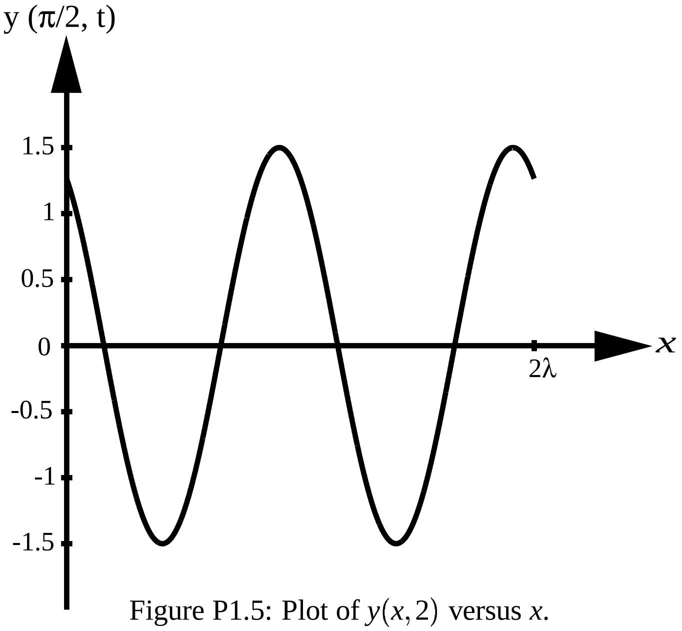


Figure P1.5: Plot of $y(x,2)$ versus x.

At $t = 2$ s, $y(x,2) = 1.5\sin(1-0.6x)$ (m), with the argument of the cosine function given in radians. Plot is shown in Fig. P1.5.

Problem 1.6 A wave traveling along a string in the $+x$-direction is given by

$$y_1(x,t) = A\cos(\omega t - \beta x),$$

where $x = 0$ is the end of the string, which is tied rigidly to a wall, as shown in Fig. 1-21 (P1.6). When wave $y_1(x,t)$ arrives at the wall, a reflected wave $y_2(x,t)$ is generated. Hence, at any location on the string, the vertical displacement y_s will be the sum of the incident and reflected waves:

$$y_s(x,t) = y_1(x,t) + y_2(x,t).$$

(a) Write down an expression for $y_2(x,t)$, keeping in mind its direction of travel and the fact that the end of the string cannot move.

(b) Generate plots of $y_1(x,t)$, $y_2(x,t)$ and $y_s(x,t)$ versus x over the range $-2\lambda \le x \le 0$ at $\omega t = \pi/4$ and at $\omega t = \pi/2$.

Solution:

(a) Since wave $y_2(x,t)$ was caused by wave $y_1(x,t)$, the two waves must have the same angular frequency ω, and since $y_2(x,t)$ is traveling on the same string as $y_1(x,t)$,

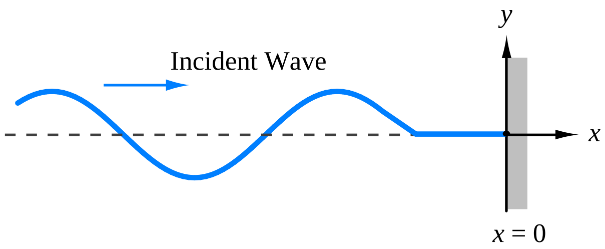


Figure P1.6: Wave on a string tied to a wall at $x = 0$ (Problem 1.6).

the two waves must have the same phase constant β. Hence, with its direction being in the negative x-direction, $y_2(x,t)$ is given by the general form

$$y_2(x,t) = B\cos(\omega t + \beta x + \phi_0), \tag{1}$$

where B and ϕ_0 are yet-to-be-determined constants. The total displacement is

$$y_s(x,t) = y_1(x,t) + y_2(x,t) = A\cos(\omega t - \beta x) + B\cos(\omega t + \beta x + \phi_0).$$

Since the string cannot move at $x = 0$, the point at which it is attached to the wall, $y_s(0,t) = 0$ for all t. Thus,

$$y_s(0,t) = A\cos\omega t + B\cos(\omega t + \phi_0) = 0. \tag{2}$$

(i) Easy Solution: The physics of the problem suggests that a possible solution for (2) is $B = -A$ and $\phi_0 = 0$, in which case we have

$$y_2(x,t) = -A\cos(\omega t + \beta x). \tag{3}$$

(ii) Rigorous Solution: By expanding the second term in (2), we have

$$A\cos\omega t + B(\cos\omega t\cos\phi_0 - \sin\omega t\sin\phi_0) = 0,$$

or

$$(A + B\cos\phi_0)\cos\omega t - (B\sin\phi_0)\sin\omega t = 0. \tag{4}$$

This equation has to be satisfied for all values of t. At $t = 0$, it gives

$$A + B\cos\phi_0 = 0, \tag{5}$$

and at $\omega t = \pi/2$, (4) gives

$$B \sin \phi_0 = 0. \tag{6}$$

Equations (5) and (6) can be satisfied simultaneously only if

$$A = B = 0 \tag{7}$$

or

$$A = -B \quad \text{and} \quad \phi_0 = 0. \tag{8}$$

Clearly (7) is not an acceptable solution because it means that $y_1(x,t) = 0$, which is contrary to the statement of the problem. The solution given by (8) leads to (3).

(b) At $\omega t = \pi/4$,

$$y_1(x,t) = A\cos(\pi/4 - \beta x) = A\cos\left(\frac{\pi}{4} - \frac{2\pi x}{\lambda}\right),$$
$$y_2(x,t) = -A\cos(\omega t + \beta x) = -A\cos\left(\frac{\pi}{4} + \frac{2\pi x}{\lambda}\right).$$

Plots of y_1, y_2, and y_3 are shown in Fig. P1.6(b).

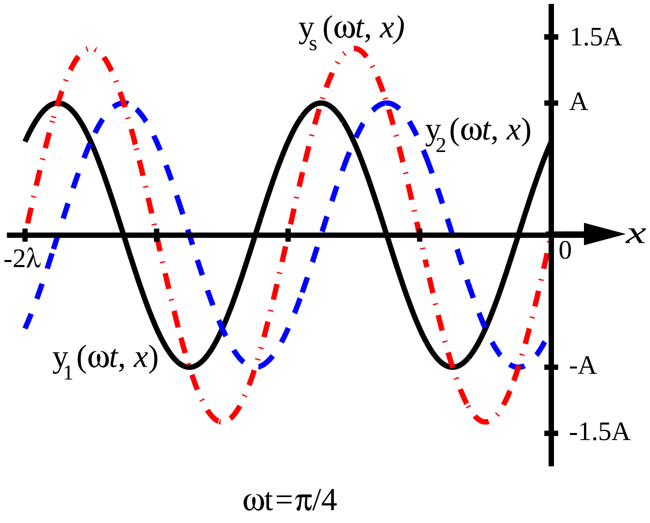


Figure P1.6: (b) Plots of y_1, y_2, and y_s versus x at $\omega t = \pi/4$.

At $\omega t = \pi/2$,

$$y_1(x,t) = A\cos(\pi/2 - \beta x) = A\sin\beta x = A\sin\frac{2\pi x}{\lambda},$$

$$y_2(x,t) = -A\cos(\pi/2 + \beta x) = A\sin\beta x = A\sin\frac{2\pi x}{\lambda}.$$

Plots of y_1, y_2, and y_3 are shown in Fig. P1.6(c).

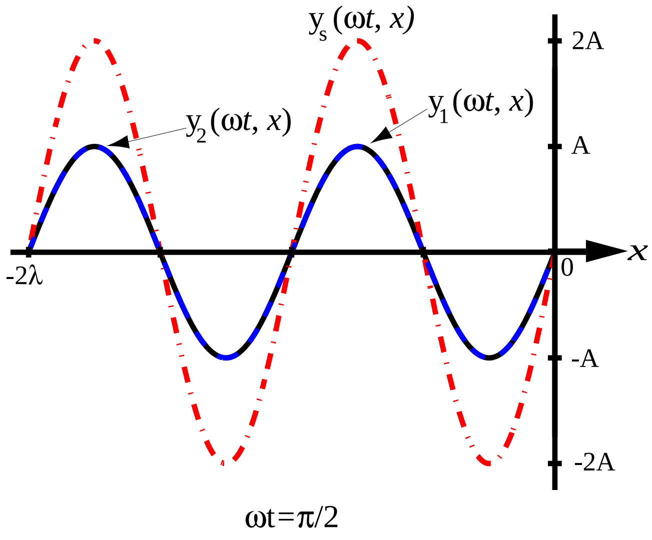


Figure P1.6: (c) Plots of y_1, y_2, and y_s versus x at $\omega t = \pi/2$.

Problem 1.7 Two waves on a string are given by the following functions:

$$y_1(x,t) = 4\cos(20t - 30x) \quad \text{(cm)},$$
$$y_2(x,t) = -4\cos(20t + 30x) \quad \text{(cm)},$$

where x is in centimeters. The waves are said to interfere constructively when their superposition $|y_s| = |y_1 + y_2|$ is a maximum and they interfere destructively when $|y_s|$ is a minimum.

(a) What are the directions of propagation of waves $y_1(x,t)$ and $y_2(x,t)$?

(b) At $t = (\pi/50)$ s, at what location x do the two waves interfere constructively, and what is the corresponding value of $|y_s|$?

(c) At $t = (\pi/50)$ s, at what location x do the two waves interfere destructively, and what is the corresponding value of $|y_s|$?

Solution:

(a) $y_1(x,t)$ is traveling in positive x-direction. $y_2(x,t)$ is traveling in negative x-direction.

(b) At $t = (\pi/50)$ s, $y_s = y_1 + y_2 = 4[\cos(0.4\pi - 30x) - \cos(0.4\pi + 3x)]$. Using the formulas from Appendix C,

$$2\sin x \sin y = \cos(x-y) - (\cos x + y),$$

we have

$$y_s = 8\sin(0.4\pi)\sin 30x = 7.61\sin 30x.$$

Hence,

$$|y_{\rm s}|_{\rm max} = 7.61$$

and it occurs when $\sin 30x = 1$, or $30x = \frac{\pi}{2} + 2n\pi$, or $x = \left(\frac{\pi}{60} + \frac{2n\pi}{30}\right)$ cm, where $n = 0, 1, 2, \ldots$.

(c) $|y_{\rm s}|_{\rm min} = 0$ and it occurs when $30x = n\pi$, or $x = \frac{n\pi}{30}$ cm.

Problem 1.8 Give expressions for $y(x,t)$ for a sinusoidal wave traveling along a string in the negative x-direction, given that $y_{\rm max} = 40$ cm, $\lambda = 30$ cm, $f = 10$ Hz, and

(a) $y(x,0) = 0$ at $x = 0$,

(b) $y(x,0) = 0$ at $x = 7.5$ cm.

Solution: For a wave traveling in the negative x-direction, we use Eq. (1.17) with $\omega = 2\pi f = 20\pi$ (rad/s), $\beta = 2\pi/\lambda = 2\pi/0.3 = 20\pi/3$ (rad/s), $A = 40$ cm, and x assigned a positive sign:

$$y(x,t) = 40\cos\left(20\pi t + \frac{20\pi}{3}x + \phi_0\right) \quad \text{(cm)},$$

with x in meters.

(a) $y(0,0) = 0 = 40\cos\phi_0$. Hence, $\phi_0 = \pm\pi/2$, and

$$\begin{aligned} y(x,t) &= 40\cos\left(20\pi t + \frac{20\pi}{3}x \pm \frac{\pi}{2}\right) \\ &= \begin{cases} -40\sin\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = \pi/2, \\ 40\sin\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = -\pi/2. \end{cases} \end{aligned}$$

(b) At $x = 7.5$ cm $= 7.5 \times 10^{-2}$ m, $y = 0 = 40\cos(\pi/2 + \phi_0)$. Hence, $\phi_0 = 0$ or π, and

$$y(x,t) = \begin{cases} 40\cos\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = 0, \\ -40\cos\left(20\pi t + \frac{20\pi}{3}x\right) \text{ (cm)}, & \text{if } \phi_0 = \pi. \end{cases}$$

Problem 1.9 An oscillator that generates a sinusoidal wave on a string completes 20 vibrations in 50 s. The wave peak is observed to travel a distance of 2.8 m along the string in 50 s. What is the wavelength?

Solution:

$$T = \frac{50}{20} = 2.5 \text{ s}, \qquad u_{\text{p}} = \frac{2.8}{5} = 0.56 \text{ m/s},$$
$$\lambda = u_{\text{p}} T = 0.56 \times 2.5 = 1.4 \text{ m}.$$

Problem 1.10 The vertical displacement of a string is given by the harmonic function:

$$y(x,t) = 6\cos(16\pi t - 20\pi x) \quad \text{(m)},$$

where x is the horizontal distance along the string in meters. Suppose a tiny particle were to be attached to the string at $x = 5$ cm, obtain an expression for the vertical velocity of the particle as a function of time.

Solution:

$$y(x,t) = 6\cos(16\pi t - 20\pi x) \quad \text{(m)}.$$

$$\begin{aligned} u(0.05,t) &= \left.\frac{dy(x,t)}{dt}\right|_{x=0.05} \\ &= 96\pi \sin(16\pi t - 20\pi x)|_{x=0.05} \\ &= 96\pi \sin(16\pi t - \pi) \\ &= -96\pi \sin(16\pi t) \quad \text{(m/s)}. \end{aligned}$$

Problem 1.11 Given two waves characterized by

$$\begin{aligned} y_1(t) &= 3\cos\omega t, \\ y_2(t) &= 3\sin(\omega t + 36^\circ), \end{aligned}$$

does $y_2(t)$ lead or lag $y_1(t)$, and by what phase angle?

Solution: We need to express $y_2(t)$ in terms of a cosine function:

$$\begin{aligned} y_2(t) &= 3\sin(\omega t + 36^\circ) \\ &= 3\cos\left(\frac{\pi}{2} - \omega t - 36^\circ\right) = 3\cos(54^\circ - \omega t) = 3\cos(\omega t - 54^\circ). \end{aligned}$$

Hence, $y_2(t)$ lags $y_1(t)$ by 54°.

Problem 1.12 The voltage of an electromagnetic wave traveling on a transmission line is given by $v(z,t) = 5e^{-\alpha z}\sin(4\pi \times 10^9 t - 20\pi z)$ (V), where z is the distance in meters from the generator.

(a) Find the frequency, wavelength, and phase velocity of the wave.

(b) At $z = 2$ m, the amplitude of the wave was measured to be 1 V. Find α.

Solution:

(a) This equation is similar to that of Eq. (1.28) with $\omega = 4\pi \times 10^9$ rad/s and $\beta = 20\pi$ rad/m. From Eq. (1.29a), $f = \omega/2\pi = 2 \times 10^9$ Hz $= 2$ GHz; from Eq. (1.29b), $\lambda = 2\pi/\beta = 0.1$ m. From Eq. (1.30),

$$u_\text{p} = \omega/\beta = 2 \times 10^8 \text{ m/s}.$$

(b) Using just the amplitude of the wave,

$$1 = 5e^{-\alpha 2}, \qquad \alpha = \frac{-1}{2\text{ m}} \ln\left(\frac{1}{5}\right) = 0.81 \text{ Np/m}.$$

Problem 1.13 A certain electromagnetic wave traveling in sea water was observed to have an amplitude of 98.02 (V/m) at a depth of 10 m and an amplitude of 81.87 (V/m) at a depth of 100 m. What is the attenuation constant of sea water?

Solution: The amplitude has the form $Ae^{\alpha z}$. At $z = 10$ m,

$$Ae^{-10\alpha} = 98.02$$

and at $z = 100$ m,

$$Ae^{-100\alpha} = 81.87$$

The ratio gives

$$\frac{e^{-10\alpha}}{e^{-100\alpha}} = \frac{98.02}{81.87} = 1.20$$

or

$$e^{-10\alpha} = 1.2e^{-100\alpha}.$$

Taking the natural log of both sides gives

$$\begin{aligned} \ln(e^{-10\alpha}) &= \ln(1.2e^{-100\alpha}), \\ -10\alpha &= \ln(1.2) - 100\alpha, \\ 90\alpha &= \ln(1.2) = 0.18. \end{aligned}$$

Hence,

$$\alpha = \frac{0.18}{90} = 2 \times 10^{-3} \quad \text{(Np/m)}.$$

Section 1-5: Complex Numbers

Problem 1.14 Evaluate each of the following complex numbers and express the result in rectangular form:

(a) $z_1 = 4e^{j\pi/3}$,
(b) $z_2 = \sqrt{3}\, e^{j3\pi/4}$,
(c) $z_3 = 6e^{-j\pi/2}$,
(d) $z_4 = j^3$,
(e) $z_5 = j^{-4}$,
(f) $z_6 = (1-j)^3$,
(g) $z_7 = (1-j)^{1/2}$.

Solution: (Note: In the following solutions, numbers are expressed to only two decimal places, but the final answers are found using a calculator with 10 decimal places.)

(a) $z_1 = 4e^{j\pi/3} = 4(\cos\pi/3 + j\sin\pi/3) = 2.0 + j3.46$.

(b)

$$z_2 = \sqrt{3}e^{j3\pi/4} = \sqrt{3}\left[\cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right)\right] = -1.22 + j1.22 = 1.22(-1+j).$$

(c) $z_3 = 6e^{-j\pi/2} = 6[\cos(-\pi/2) + j\sin(-\pi/2)] = -j6$.

(d) $z_4 = j^3 = j \cdot j^2 = -j$, or

$$z_4 = j^3 = (e^{j\pi/2})^3 = e^{j3\pi/2} = \cos(3\pi/2) + j\sin(3\pi/2) = -j.$$

(e) $z_5 = j^{-4} = (e^{j\pi/2})^{-4} = e^{-j2\pi} = 1$.

(f)

$$\begin{aligned} z_6 = (1-j)^3 = (\sqrt{2}e^{-j\pi/4})^3 &= (\sqrt{2})^3 e^{-j3\pi/4} \\ &= (\sqrt{2})^3[\cos(3\pi/4) - j\sin(3\pi/4)] \\ &= -2 - j2 = -2(1+j). \end{aligned}$$

(g)

$$\begin{aligned} z_7 = (1-j)^{1/2} = (\sqrt{2}e^{-j\pi/4})^{1/2} = \pm 2^{1/4}e^{-j\pi/8} &= \pm 1.19(0.92 - j0.38) \\ &= \pm(1.10 - j0.45). \end{aligned}$$

Problem 1.15 Complex numbers z_1 and z_2 are given by

$$\begin{aligned} z_1 &= 3 - j2, \\ z_2 &= -4 + j3. \end{aligned}$$

(a) Express z_1 and z_2 in polar form.
(b) Find $|z_1|$ by applying Eq. (1.41) and again by applying Eq. (1.43).
(c) Determine the product $z_1 z_2$ in polar form.
(d) Determine the ratio z_1/z_2 in polar form.
(e) Determine z_1^3 in polar form.

Solution:
(a) Using Eq. (1.41),

$$\begin{aligned} z_1 &= 3 - j2 = 3.6e^{-j33.7^\circ}, \\ z_2 &= -4 + j3 = 5e^{j143.1^\circ}. \end{aligned}$$

(b) By Eq. (1.41) and Eq. (1.43), respectively,

$$\begin{aligned} |z_1| &= |3 - j2| = \sqrt{3^2 + (-2)^2} = \sqrt{13} = 3.60, \\ |z_1| &= \sqrt{(3 - j2)(3 + j2)} = \sqrt{13} = 3.60. \end{aligned}$$

(c) By applying Eq. (1.47b) to the results of part (a),

$$z_1 z_2 = 3.6e^{-j33.7^\circ} \times 5e^{j143.1^\circ} = 18e^{j109.4^\circ}.$$

(d) By applying Eq. (1.48b) to the results of part (a),

$$\frac{z_1}{z_2} = \frac{3.6e^{-j33.7^\circ}}{5e^{j143.1^\circ}} = 0.72e^{-j176.8^\circ}.$$

(e) By applying Eq. (1.49) to the results of part (a),

$$z_1^3 = \left(3.6e^{-j33.7^\circ}\right)^3 = (3.6)^3 e^{-j3\times 33.7^\circ} = 46.66e^{-j101.1^\circ}.$$

Problem 1.16 If $z = -2 + j4$, determine the following quantities in polar form:
(a) $1/z$,
(b) z^3,
(c) $|z|^2$,
(d) $\mathfrak{Im}\{z\}$,
(e) $\mathfrak{Im}\{z^*\}$.

Solution: (Note: In the following solutions, numbers are expressed to only two decimal places, but the final answers are found using a calculator with 10 decimal places.)

(a)

$$\frac{1}{z} = \frac{1}{-2+j4} = (-2+j4)^{-1} = (4.47\,e^{j116.6^\circ})^{-1} = (4.47)^{-1}e^{-j116.6^\circ} = 0.22\,e^{-j116.6^\circ}.$$

(b) $z^3 = (-2+j4)^3 = (4.47\,e^{j116.6^\circ})^3 = (4.47)^3 e^{j350.0^\circ} = 89.44\,e^{-j10^\circ}$.
(c) $|z|^2 = z \cdot z^* = (-2+j4)(-2-j4) = 4+16 = 20$.
(d) $\mathfrak{Im}\{z\} = \mathfrak{Im}\{-2+j4\} = 4$.
(e) $\mathfrak{Im}\{z^*\} = \mathfrak{Im}\{-2-j4\} = -4 = 4e^{j\pi}$.

Problem 1.17 Find complex numbers $t = z_1 + z_2$ and $s = z_1 - z_2$, both in polar form, for each of the following pairs:
(a) $z_1 = 2+j3,\ z_2 = 1-j3$,
(b) $z_1 = 3,\ z_2 = -j3$,
(c) $z_1 = 3\angle 30^\circ,\ z_2 = 3\angle -30^\circ$,
(d) $z_1 = 3\angle 30^\circ,\ z_2 = 3\angle -150^\circ$.

Solution:
(a)

$$\begin{aligned} t &= z_1 + z_2 = (2+j3) + (1-j3) = 3, \\ s &= z_1 - z_2 = (2+j3) - (1-j3) = 1 + j6 = 6.08\,e^{j80.5^\circ}. \end{aligned}$$

(b)

$$\begin{aligned} t &= z_1 + z_2 = 3 - j3 = 4.24\,e^{-j45^\circ}, \\ s &= z_1 - z_2 = 3 + j3 = 4.24\,e^{j45^\circ}. \end{aligned}$$

(c)

$$\begin{aligned} t &= z_1 + z_2 = 3\angle 30^\circ + 3\angle -30^\circ \\ &= 3e^{j30^\circ} + 3e^{-j30^\circ} = (2.6 + j1.5) + (2.6 - j1.5) = 5.2, \\ s &= z_1 - z_2 = 3e^{j30^\circ} - 3e^{-j30^\circ} = (2.6 + j1.5) - (2.6 - j1.5) = j3 = 3e^{j90^\circ}. \end{aligned}$$

(d)

$$\begin{aligned} t &= z_1 + z_2 = 3\angle 30^\circ + 3\angle -150^\circ = (2.6 + j1.5) + (-2.6 - j1.5) = 0, \\ s &= z_1 - z_2 = (2.6 + j1.5) - (-2.6 - j1.5) = 5.2 + j3 = 6e^{j30^\circ}. \end{aligned}$$

Problem 1.18 Complex numbers z_1 and z_2 are given by

$$\begin{aligned} z_1 &= 5\angle -60^\circ, \\ z_2 &= 2\angle 45^\circ. \end{aligned}$$

(a) Determine the product $z_1 z_2$ in polar form.
(b) Determine the product $z_1 z_2^*$ in polar form.
(c) Determine the ratio z_1/z_2 in polar form.
(d) Determine the ratio z_1^*/z_2^* in polar form.
(e) Determine $\sqrt{z_1}$ in polar form.

Solution:

(a) $z_1 z_2 = 5e^{-j60^\circ} \times 2e^{j45^\circ} = 10e^{-j15^\circ}$.
(b) $z_1 z_2^* = 5e^{-j60^\circ} \times 2e^{-j45^\circ} = 10e^{-j105^\circ}$.
(c) $\dfrac{z_1}{z_2} = \dfrac{5e^{-j60^\circ}}{2e^{j45^\circ}} = 2.5^{-j105^\circ}$.
(d) $\dfrac{z_1^*}{z_2^*} = \left(\dfrac{z_1}{z_2}\right)^* = 2.5^{j105^\circ}$.
(e) $\sqrt{z_1} = \sqrt{5e^{-j60^\circ}} = \pm\sqrt{5}\,e^{-j30^\circ}$.

Problem 1.19 If $z = 3 - j5$, find the value of $\ln(z)$.

Solution:

$$
\begin{aligned}
|z| &= +\sqrt{3^2 + 5^2} = 5.83, \quad \theta = \tan^{-1}\left(\frac{-5}{3}\right) = -59^\circ, \\
z &= |z|e^{j\theta} = 5.83e^{-j59^\circ}, \\
\ln(z) &= \ln(5.83e^{-j59^\circ}) \\
&= \ln(5.83) + \ln(e^{-j59^\circ}) \\
&= 1.76 - j59^\circ = 1.76 - j\frac{59^\circ \pi}{180^\circ} = 1.76 - j1.03.
\end{aligned}
$$

Problem 1.20 If $z = 3 - j4$, find the value of e^z.

Solution:

$$
\begin{aligned}
&e^z = e^{3-j4} = e^3 \cdot e^{-j4} = e^3(\cos 4 - j\sin 4), \\
&e^3 = 20.09, \qquad \text{and} \qquad 4 \text{ rad} = \frac{4}{\pi} \times 180^\circ = 229.18^\circ.
\end{aligned}
$$

Hence, $e^z = 20.08(\cos 229.18^\circ - j\sin 229.18^\circ) = -13.13 + j15.20$.

Section 1-6: Phasors

Problem 1.21 A voltage source given by $v_s(t) = 25\cos(2\pi \times 10^3 t - 30°)$ (V) is connected to a series RC load as shown in Fig. 1-19. If $R = 1$ MΩ and $C = 200$ pF, obtain an expression for $v_c(t)$, the voltage across the capacitor.

Solution: In the phasor domain, the circuit is a voltage divider, and

$$\widetilde{V}_c = \widetilde{V}_s \frac{1/j\omega C}{R + 1/j\omega C} = \frac{\widetilde{V}_s}{(1 + j\omega RC)} .$$

Now $\widetilde{V}_s = 25e^{-j30^\circ}$ V with $\omega = 2\pi \times 10^3$ rad/s, so

$$\begin{aligned}\widetilde{V}_c &= \frac{25e^{-j30^\circ} \text{ V}}{1 + j((2\pi \times 10^3 \text{ rad/s}) \times (10^6\ \Omega) \times (200 \times 10^{-12} \text{ F}))} \\ &= \frac{25e^{-j30^\circ} \text{ V}}{1 + j2\pi/5} = 15.57e^{-j81.5^\circ} \text{ V}.\end{aligned}$$

Converting back to an instantaneous value,

$$v_c(t) = \mathfrak{Re}\,\widetilde{V}_c e^{j\omega t} = \mathfrak{Re}\,15.57e^{j(\omega t - 81.5^\circ)} \text{ V} = 15.57\cos(2\pi \times 10^3 t - 81.5^\circ) \text{ V},$$

where t is expressed in seconds.

Problem 1.22 Find the phasors of the following time functions:

(a) $v(t) = 3\cos(\omega t - \pi/3)$ (V),
(b) $v(t) = 12\sin(\omega t + \pi/4)$ (V),
(c) $i(x,t) = 2e^{-3x}\sin(\omega t + \pi/6)$ (A),
(d) $i(t) = -2\cos(\omega t + 3\pi/4)$ (A),
(e) $i(t) = 4\sin(\omega t + \pi/3) + 3\cos(\omega t - \pi/6)$ (A).

Solution:

(a) $\widetilde{V} = 3e^{-j\pi/3}$ V.

(b) $v(t) = 12\sin(\omega t + \pi/4) = 12\cos(\pi/2 - (\omega t + \pi/4)) = 12\cos(\omega t - \pi/4)$ V, $\widetilde{V} = 12e^{-j\pi/4}$ V.

(c)

$$\begin{aligned}i(t) &= 2e^{-3x}\sin(\omega t + \pi/6) \text{ A} = 2e^{-3x}\cos(\pi/2 - (\omega t + \pi/6)) \text{ A} \\ &= 2e^{-3x}\cos(\omega t - \pi/3) \text{ A}, \\ \widetilde{I} &= 2e^{-3x}e^{-j\pi/3} \text{ A}.\end{aligned}$$

(d)

$$\begin{aligned} i(t) &= -2\cos(\omega t + 3\pi/4), \\ \widetilde{I} &= -2e^{j3\pi/4} = 2e^{-j\pi}e^{j3\pi/4} = 2e^{-j\pi/4} \text{ A}. \end{aligned}$$

(e)

$$\begin{aligned} i(t) &= 4\sin(\omega t + \pi/3) + 3\cos(\omega t - \pi/6) \\ &= 4\cos[\pi/2 - (\omega t + \pi/3)] + 3\cos(\omega t - \pi/6) \\ &= 4\cos(-\omega t + \pi/6) + 3\cos(\omega t - \pi/6) \\ &= 4\cos(\omega t - \pi/6) + 3\cos(\omega t - \pi/6) = 7\cos(\omega t - \pi/6), \\ \widetilde{I} &= 7e^{-j\pi/6} \text{ A}. \end{aligned}$$

Problem 1.23 Find the instantaneous time sinusoidal functions corresponding to the following phasors:

(a) $\widetilde{V} = -5e^{j\pi/3}$ (V),
(b) $\widetilde{V} = j6e^{-j\pi/4}$ (V),
(c) $\widetilde{I} = (6 + j8)$ (A),
(d) $\tilde{I} = -3 + j2$ (A),
(e) $\tilde{I} = j$ (A),
(f) $\tilde{I} = 2e^{j\pi/6}$ (A).

Solution:

(a)

$$\begin{aligned} \widetilde{V} &= -5e^{j\pi/3} \text{ V} = 5e^{j(\pi/3-\pi)} \text{ V} = 5e^{-j2\pi/3} \text{ V}, \\ v(t) &= 5\cos(\omega t - 2\pi/3) \text{ V}. \end{aligned}$$

(b)

$$\begin{aligned} \widetilde{V} &= j6e^{-j\pi/4} \text{ V} = 6e^{j(-\pi/4+\pi/2)} \text{ V} = 6e^{j\pi/4} \text{ V}, \\ v(t) &= 6\cos(\omega t + \pi/4) \text{ V}. \end{aligned}$$

(c)

$$\begin{aligned} \widetilde{I} &= (6 + j8) \text{ A} = 10e^{j53.1^\circ} \text{ A}, \\ i(t) &= 10\cos(\omega t + 53.1^\circ) \text{ A}. \end{aligned}$$

(d)

$$\begin{aligned} \widetilde{I} &= -3 + j2 = 3.61\,e^{j146.31^\circ}, \\ i(t) &= \mathfrak{Re}\{3.61\,e^{j146.31^\circ}\,e^{j\omega t}\} = 3.61\,\cos(\omega t + 146.31^\circ) \text{ A}. \end{aligned}$$

(e)

$$\widetilde{I} = j = e^{j\pi/2},$$
$$i(t) = \mathfrak{Re}\{e^{j\pi/2}e^{j\omega t}\} = \cos(\omega t + \pi/2) = -\sin\omega t \text{ A}.$$

(f)

$$\widetilde{I} = 2e^{j\pi/6},$$
$$i(t) = \mathfrak{Re}\{2e^{j\pi/6}e^{j\omega t}\} = 2\cos(\omega t + \pi/6) \text{ A}.$$

Problem 1.24 A series RLC circuit is connected to a generator with a voltage $v_s(t) = V_0\cos(\omega t + \pi/3)$ (V).

(a) Write down the voltage loop equation in terms of the current $i(t)$, R, L, C, and $v_s(t)$.

(b) Obtain the corresponding phasor-domain equation.

(c) Solve the equation to obtain an expression for the phasor current $\widetilde{I}$.

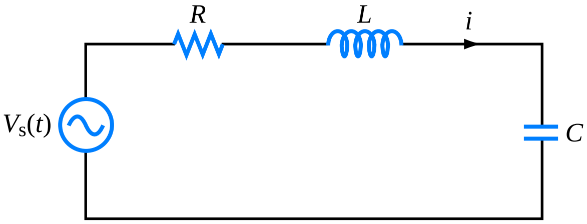


Figure P1.24: RLC circuit.

Solution:

(a) $v_s(t) = Ri + L\dfrac{di}{dt} + \dfrac{1}{C}\displaystyle\int i\,dt.$

(b) In phasor domain: $\widetilde{V}_s = R\widetilde{I} + j\omega L\widetilde{I} + \dfrac{\widetilde{I}}{j\omega C}$.

(c) $\widetilde{I} = \dfrac{\widetilde{V}_s}{R + j(\omega L - 1/\omega C)} = \dfrac{V_0 e^{j\pi/3}}{R + j(\omega L - 1/\omega C)} = \dfrac{\omega C V_0 e^{j\pi/3}}{\omega RC + j(\omega^2 LC - 1)}$.

Problem 1.25 A wave traveling along a string is given by

$$y(x,t) = 2\sin(4\pi t + 10\pi x) \quad \text{(cm)}$$

where x is the distance along the string in meters and y is the vertical displacement. Determine: **(a)** the direction of wave travel, **(b)** the reference phase ϕ_0, **(c)** the frequency, **(d)** the wavelength, and **(e)** the phase velocity.

Solution:

(a) We start by converting the given expression into a cosine function of the form given by (1.17):

$$y(x,t) = 2\cos\left(4\pi t + 10\pi x - \frac{\pi}{2}\right) \quad \text{(cm)}.$$

Since the coefficients of t and x both have the same sign, the wave is traveling in the negative x-direction.

(b) From the cosine expression, $\phi_0 = -\pi/2$.

(c) $\omega = 2\pi f = 4\pi$,

$$f = 4\pi/2\pi = 2 \text{ Hz}.$$

(d) $2\pi/\lambda = 10\pi$,

$$\lambda = 2\pi/10\pi = 0.2 \text{ m}.$$

(e) $u_\text{p} = f\lambda = 2 \times 0.2 = 0.4$ (m/s).

Problem 1.26 A laser beam traveling through fog was observed to have an intensity of 1 (μW/m^2) at a distance of 2 m from the laser gun and an intensity of 0.2 (μW/m^2) at a distance of 3 m. Given that the intensity of an electromagnetic wave is proportional to the square of its electric-field amplitude, find the attenuation constant α of fog.

Solution: If the electric field is of the form

$$E(x,t) = E_0 e^{-\alpha x}\cos(\omega t - \beta x),$$

then the intensity must have a form

$$\begin{aligned} I(x,t) &\approx [E_0 e^{-\alpha x}\cos(\omega t - \beta x)]^2 \\ &\approx E_0^2 e^{-2\alpha x}\cos^2(\omega t - \beta x) \end{aligned}$$

or

$$I(x,t) = I_0 e^{-2\alpha x}\cos^2(\omega t - \beta x)$$

where we define $I_0 \approx E_0^2$. We observe that the magnitude of the intensity varies as $I_0 e^{-2\alpha x}$. Hence,

$$\begin{aligned} &\text{at } x = 2 \text{ m}, \qquad I_0 e^{-4\alpha} = 1 \times 10^{-6} \quad (\text{W/m}^2), \\ &\text{at } x = 3 \text{ m}, \qquad I_0 e^{-6\alpha} = 0.2 \times 10^{-6} \quad (\text{W/m}^2). \end{aligned}$$

$$\begin{aligned}\frac{I_0e^{-4\alpha}}{I_0e^{-6\alpha}} &= \frac{10^{-6}}{0.2\times10^{-6}} = 5\\ e^{-4\alpha}\cdot e^{6\alpha} &= e^{2\alpha} = 5\\ \alpha &= 0.8 \quad \text{(NP/m).}\end{aligned}$$

Problem 1.27 Complex numbers z_1 and z_2 are given by

$$\begin{aligned}z_1 &= -3+j2\\ z_2 &= 1-j2\end{aligned}$$

Determine **(a)** z_1z_2, **(b)** z_1/z_2^*, **(c)** z_1^2, and **(d)** $z_1z_1^*$, all all in polar form.

Solution:

(a) We first convert z_1 and z_2 to polar form:

$$\begin{aligned}z_1 = -(3-j2) &= -\left(\sqrt{3^2+2^2}\,e^{-j\tan^{-1}2/3}\right)\\ &= -\sqrt{13}\,e^{-j33.7^\circ}\\ &= \sqrt{13}\,e^{j(180^\circ-33.7^\circ)}\\ &= \sqrt{13}\,e^{j146.3^\circ}.\end{aligned}$$

$$\begin{aligned}z_2 = 1-j2 &= \sqrt{1+4}\,e^{-j\tan^{-1}2}\\ &= \sqrt{5}\,e^{-j63.4^\circ}.\end{aligned}$$

$$\begin{aligned}z_1z_2 &= \sqrt{13}\,e^{j146.3^\circ}\times\sqrt{5}\,e^{-j63.4^\circ}\\ &= \sqrt{65}\,e^{j82.9^\circ}.\end{aligned}$$

(b)

$$\frac{z_1}{z_2^*} = \frac{\sqrt{13}\,e^{j146.3^\circ}}{\sqrt{5}\,e^{j63.4^\circ}} = \sqrt{\frac{13}{5}}\,e^{j82.9^\circ}.$$

(c)

$$\begin{aligned}z_1^2 = (\sqrt{13})^2(e^{j146.3^\circ})^2 &= 13e^{j292.6^\circ}\\ &= 13e^{-j360^\circ}e^{j292.6^\circ}\\ &= 13e^{-j67.4^\circ}.\end{aligned}$$

(d)

$$\begin{aligned} z_1 z_1^* &= \sqrt{13}\, e^{j146.3^\circ} \times \sqrt{13}\, e^{-j146.3^\circ} \\ &= 13. \end{aligned}$$

Problem 1.28 If $z = 3e^{j\pi/6}$, find the value of e^z.

Solution:

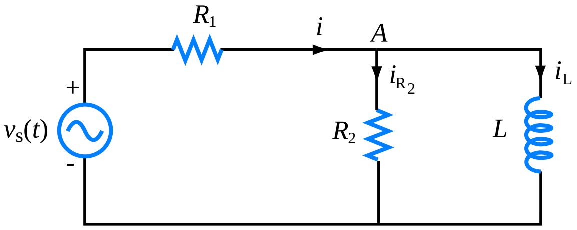
$$\begin{aligned} z = 3e^{j\pi/6} &= 3\cos\pi/6 + j3\sin\pi/6 \\ &= 2.6 + j1.5 \end{aligned}$$

$$\begin{aligned} e^z = e^{2.6+j1.5} &= e^{2.6} \times e^{j1.5} \\ &= e^{2.6}(\cos 1.5 + j\sin 1.5) \\ &= 13.46(0.07 + j0.98) \\ &= 0.95 + j13.43. \end{aligned}$$

Problem 1.29 The voltage source of the circuit shown in the figure is given by

$$v_s(t) = 25\cos(4\times 10^4 t - 45^\circ) \quad \text{(V)}.$$

Obtain an expression for $i_L(t)$, the current flowing through the inductor.



R_1 = 20 Ω, R_2 = 30 Ω, L = 0.4 mH

Solution: Based on the given voltage expression, the phasor source voltage is

$$\widetilde{V}_s = 25e^{-j45^\circ} \quad \text{(V)}. \tag{9}$$

The voltage equation for the left-hand side loop is

$$R_1 i + R_2 i_{R_2} = v_s \tag{10}$$

For the right-hand loop,

$$R_2 i_{R_2} = L\frac{di_\mathrm{L}}{dt} \ , \tag{11}$$

and at node A,

$$i = i_{R_2} + i_\mathrm{L}. \tag{12}$$

Next, we convert Eqs. (2)–(4) into phasor form:

$$\begin{aligned} R_1\widetilde{I} + R_2\widetilde{I}_{R_2} &= \widetilde{V}_\mathrm{s} && (13)\\ R_2\widetilde{I}_{R_2} &= \ j\omega L\widetilde{I}_\mathrm{L} && (14)\\ \widetilde{I} &= \widetilde{I}_{R_2} + \widetilde{I}_\mathrm{L} && (15) \end{aligned}$$

Upon combining (6) and (7) to solve for $\widetilde{I}_{R_2}$ in terms of $\widetilde{I}$, we have:

$$\widetilde{I}_{R_2} = \frac{j\omega L}{R_2 + j\omega L} I \ . \tag{16}$$

Substituting (8) in (5) and then solving for $\widetilde{I}$ leads to:

$$\begin{aligned} R_1\widetilde{I} + \frac{jR_2\omega L}{R_2 + j\omega L}\widetilde{I} &= \widetilde{V}_\mathrm{s}\\ \widetilde{I}\left(R_1 + \frac{jR_2\omega L}{R_2 + j\omega L}\right) &= \widetilde{V}_\mathrm{s}\\ \widetilde{I}\left(\frac{R_1R_2 + jR_1\omega L + jR_2\omega L}{R_2 + j\omega L}\right) &= \widetilde{V}_\mathrm{s}\\ \widetilde{I} &= \left(\frac{R_2 + j\omega L}{R_1R_2 + j\omega L(R_1 + R_2)}\right)\widetilde{V}_\mathrm{s}. && (17) \end{aligned}$$

Combining (6) and (7) to solve for $\widetilde{I}_\mathrm{L}$ in terms of $\widetilde{I}$ gives

$$\widetilde{I}_\mathrm{L} = \frac{R_2}{R_2 + j\omega L}\widetilde{I}. \tag{18}$$

Combining (9) and (10) leads to

$$\begin{aligned} \widetilde{I}_\mathrm{L} &= \left(\frac{R_2}{R_2 + j\omega L}\right)\left(\frac{R_2 + j\omega L}{R_1R_2 + j\omega L(R_1 + R_2)}\right)\widetilde{V}_\mathrm{s}\\ &= \frac{R_2}{R_1R_2 + + j\omega L(R_1 + R_2)}\widetilde{V}_\mathrm{s}. \end{aligned}$$

Using (1) for $\widetilde{V}_{\rm s}$ and replacing R_1, R_2, L and ω with their numerical values, we have

$$\begin{aligned}\widetilde{I}_{\rm L} &= \frac{30}{20\times 30 + j4\times 10^4\times 0.4\times 10^{-3}(20+30)}\ 25e^{-j45^\circ} \\ &= \frac{30\times 25}{600 + j800}\ e^{-j45^\circ} \\ &= \frac{7.5}{6+j8}\ e^{-j45^\circ} = \frac{7.5e^{-j45^\circ}}{10e^{j53.1^\circ}} = 0.75e^{-j98.1^\circ} \quad \text{(A)}.\end{aligned}$$

Finally,

$$\begin{aligned}i_{\rm L}(t) &= \mathfrak{Re}[\widetilde{I}_{\rm L}e^{j\omega t}] \\ &= 0.75\cos(4\times 10^4 t - 98.1^\circ) \quad \text{(A)}.\end{aligned}$$

Chapter 2: Transmission Lines

Lesson #4

Chapter — Section: 2-1, 2-2

Topics: Lumped-element model

Highlights:

- TEM lines
- General properties of transmission lines
- L, C, R, G

Lesson #5

Chapter — Section: 2-3, 2-4

Topics: Transmission-line equations, wave propagation

Highlights:

- Wave equation
- Characteristic impedance
- General solution

Special Illustrations:

- Example 2-1

Lesson #6

Chapter — Section: 2-5

Topics: Lossless line

Highlights:

- General wave propagation properties
- Reflection coefficient
- Standing waves
- Maxima and minima

Special Illustrations:

- Example 2-2
- Example 2-5

Lesson #7

Chapter — Section: 2-6

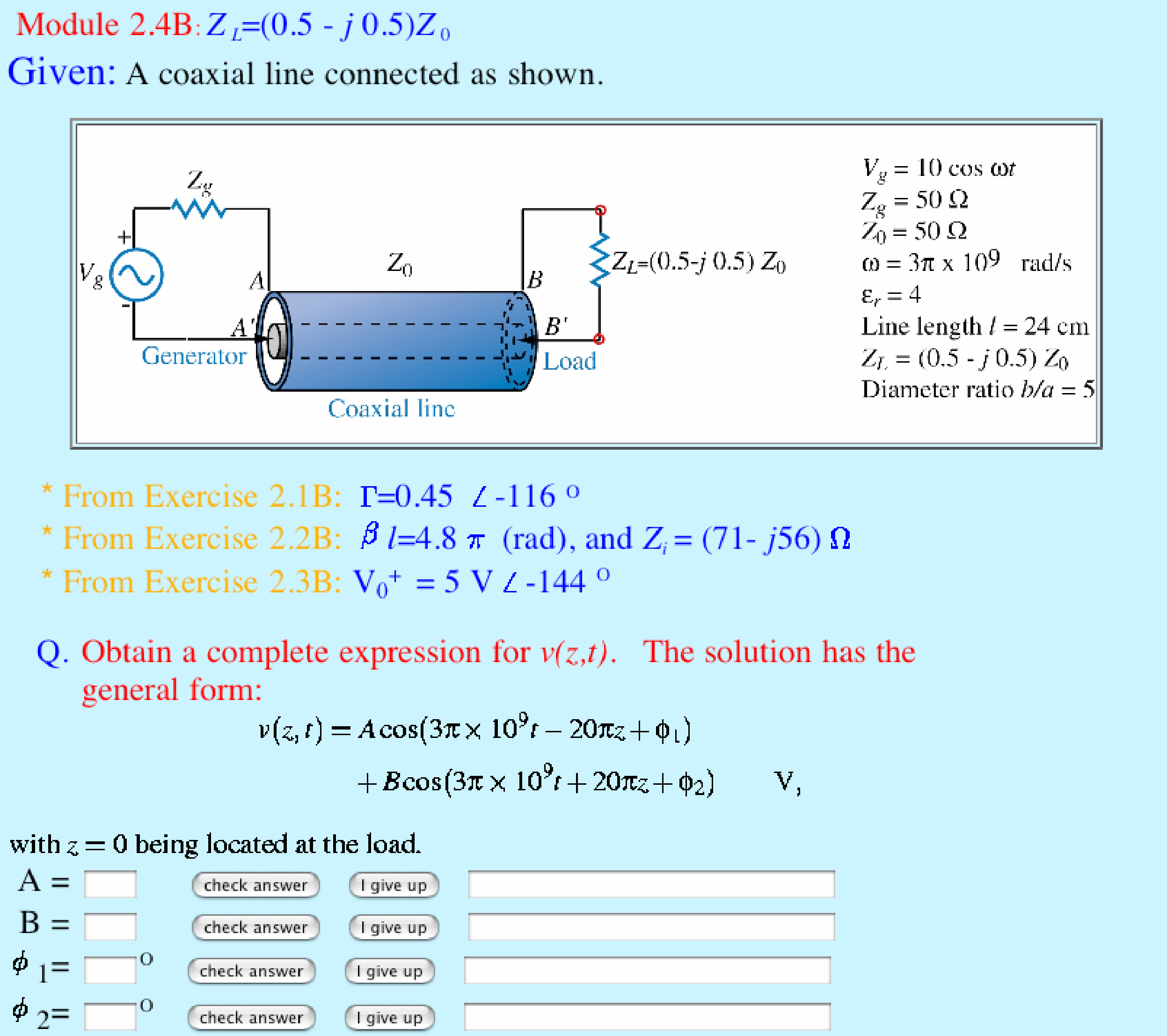
Topics: Input impedance

Highlights:

- Thévenin equivalent
- Solution for V and I at any location

Special Illustrations:

- Example 2-6
- CD-ROM Modules 2.1-2.4, Configurations A-C
- CD-ROM Demos 2.1-2.4, Configurations A-C



Module 2.4B: $Z_L=(0.5 - j\,0.5)Z_0$

Given: A coaxial line connected as shown.

$V_g = 10 \cos \omega t$
$Z_g = 50\ \Omega$
$Z_0 = 50\ \Omega$
$\omega = 3\pi \times 10^9$ rad/s
$\varepsilon_r = 4$
Line length $l = 24$ cm
$Z_L = (0.5 - j\,0.5)\,Z_0$
Diameter ratio $b/a = 5$

* From Exercise 2.1B: $\Gamma = 0.45\ \angle -116^{\circ}$

* From Exercise 2.2B: $\beta\, l = 4.8\,\pi$ (rad), and $Z_i = (71 - j56)\ \Omega$

* From Exercise 2.3B: $V_0^+ = 5\ \text{V} \angle -144^{\circ}$

Q. Obtain a complete expression for $v(z,t)$. The solution has the general form:

$$v(z,t) = A\cos(3\pi \times 10^9 t - 20\pi z + \phi_1) + B\cos(3\pi \times 10^9 t + 20\pi z + \phi_2) \qquad \text{V},$$

with $z = 0$ being located at the load.

A = ☐ check answer I give up ☐
B = ☐ check answer I give up ☐
$\phi_1 =$ ☐ $^{\circ}$ check answer I give up ☐
$\phi_2 =$ ☐ $^{\circ}$ check answer I give up ☐

Lessons #8 and 9

Chapter — Section: 2-7, 2-8

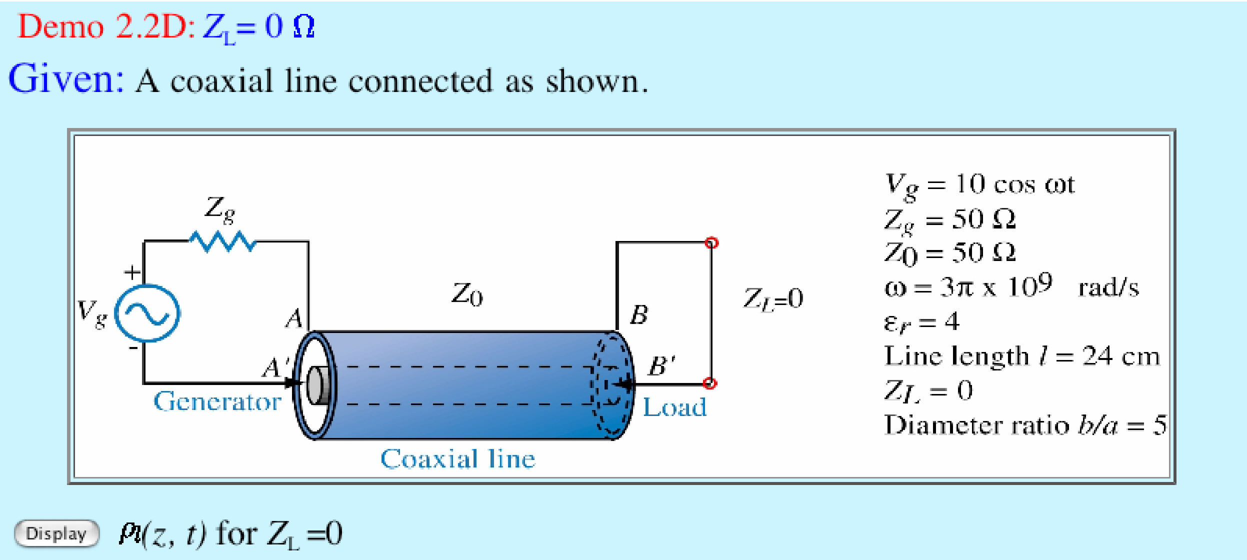
Topics: Special cases, power flow

Highlights:

- Sorted line
- Open line
- Matched line
- Quarter-wave transformer
- Power flow

Special Illustrations:

- Example 2-8
- CD-ROM Modules 2.1-2.4, Configurations D and E
- CD-ROM Demos 2.1-2.4, Configurations D and E



Demo 2.2D: $Z_L = 0\ \Omega$

Given: A coaxial line connected as shown.

$V_g = 10 \cos \omega t$

$Z_g = 50\ \Omega$

$Z_0 = 50\ \Omega$

$\omega = 3\pi \times 10^9$ rad/s

$\varepsilon_r = 4$

Line length $l = 24$ cm

$Z_L = 0$

Diameter ratio $b/a = 5$

Display $P(z, t)$ for $Z_L = 0$

Lessons #10 and 11

Chapter — Section: 2-9

Topics: Smith chart

Highlights:

- Structure of Smith chart
- Calculating impedances, admittances, transformations
- Locations of maxima and minima

Special Illustrations:

- Example 2-10
- Example 2-11

Lesson #12

Chapter — Section: 2-10

Topics: Matching

Highlights:

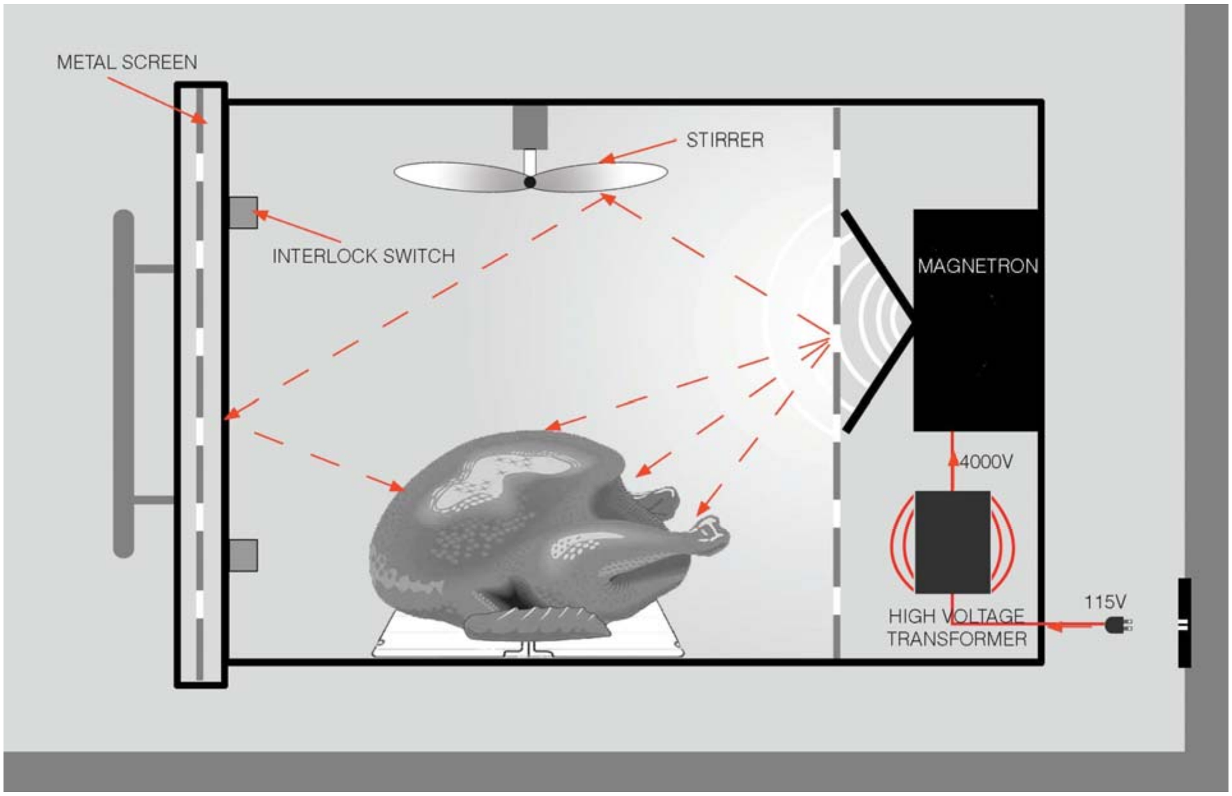
- Matching network
- Double-stub tuning

Special Illustrations:

- Example 2-12
- Technology Brief on "Microwave Oven" (CD-ROM)

Microwave Ovens

Percy Spencer, while working for Raytheon in the 1940s on the design and construction of magnetrons for radar, observed that a chocolate bar that had unintentionally been exposed to microwaves had melted in his pocket. The process of cooking by microwave was patented in 1946, and by the 1970s microwave ovens had become standard household items.



Lesson #13

Chapter — Section: 2-11

Topics: Transients

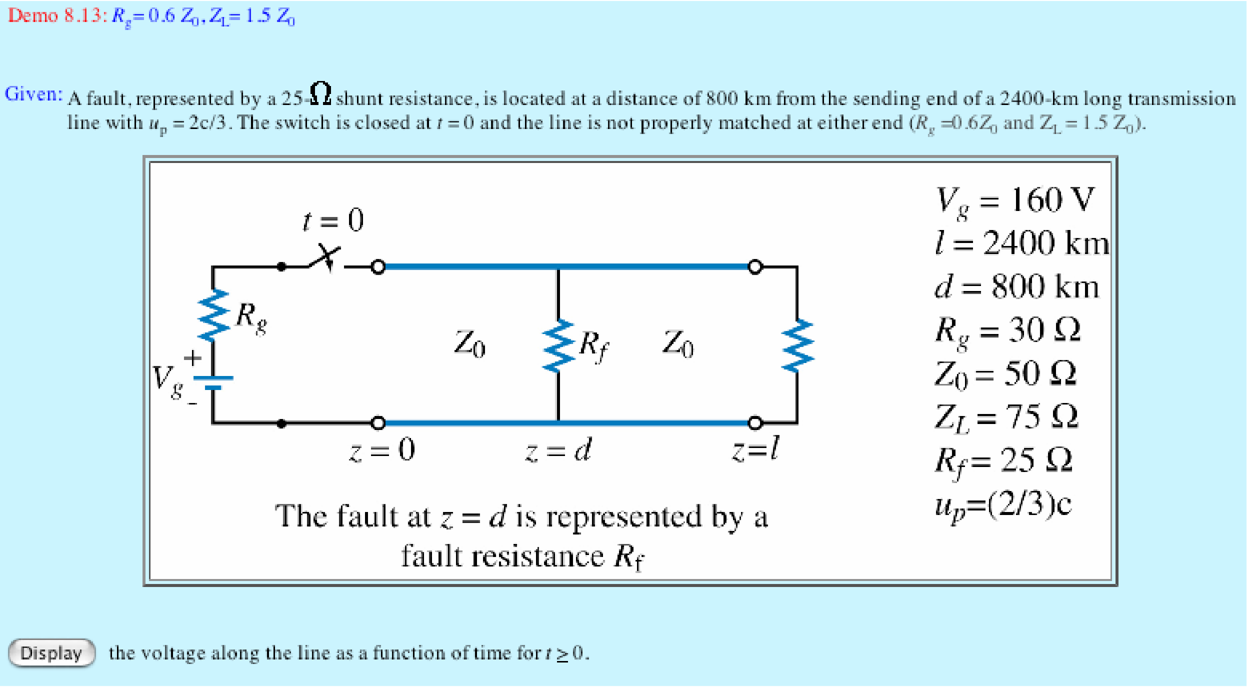
Highlights:

- Step function
- Bounce diagram

Special Illustrations:

- CD-ROM Modules 2.5-2.9
- CD-ROM Demos 2.5-2.13

Demo 2.13



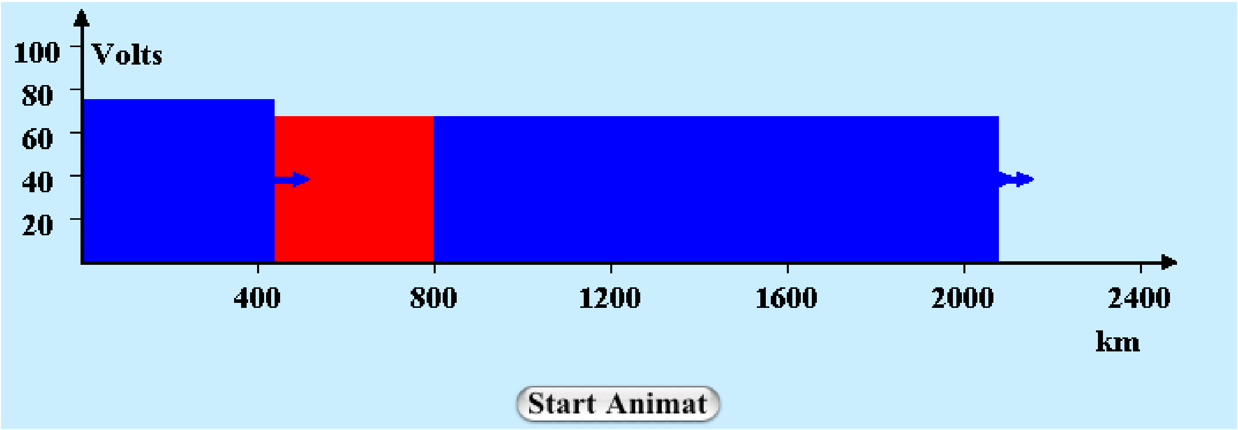
Demo 8.13: $R_g = 0.6\ Z_0$, $Z_L = 1.5\ Z_0$

Given: A fault, represented by a 25-Ω shunt resistance, is located at a distance of 800 km from the sending end of a 2400-km long transmission line with $u_p = 2c/3$. The switch is closed at $t = 0$ and the line is not properly matched at either end ($R_g = 0.6Z_0$ and $Z_L = 1.5\ Z_0$).

$V_g = 160$ V
$l = 2400$ km
$d = 800$ km
$R_g = 30\ \Omega$
$Z_0 = 50\ \Omega$
$Z_L = 75\ \Omega$
$R_f = 25\ \Omega$
$u_p = (2/3)c$

The fault at $z = d$ is represented by a fault resistance R_f

Display the voltage along the line as a function of time for $t \geq 0$.



Chapter 2

Sections 2-1 to 2-4: Transmission-Line Model

Problem 2.1 A transmission line of length l connects a load to a sinusoidal voltage source with an oscillation frequency f. Assuming the velocity of wave propagation on the line is c, for which of the following situations is it reasonable to ignore the presence of the transmission line in the solution of the circuit:

(a) $l = 20$ cm, $f = 20$ kHz,
(b) $l = 50$ km, $f = 60$ Hz,
(c) $l = 20$ cm, $f = 600$ MHz,
(d) $l = 1$ mm, $f = 100$ GHz.

Solution: A transmission line is negligible when $l/\lambda \leq 0.01$.

(a) $\dfrac{l}{\lambda} = \dfrac{lf}{u_\text{p}} = \dfrac{(20\times10^{-2}\text{ m})\times(20\times10^{3}\text{ Hz})}{3\times10^{8}\text{ m/s}} = 1.33\times10^{-5}$ (negligible).

(b) $\dfrac{l}{\lambda} = \dfrac{lf}{u_\text{p}} = \dfrac{(50\times10^{3}\text{ m})\times(60\times10^{0}\text{ Hz})}{3\times10^{8}\text{ m/s}} = 0.01$ (borderline).

(c) $\dfrac{l}{\lambda} = \dfrac{lf}{u_\text{p}} = \dfrac{(20\times10^{-2}\text{ m})\times(600\times10^{6}\text{ Hz})}{3\times10^{8}\text{ m/s}} = 0.40$ (nonnegligible).

(d) $\dfrac{l}{\lambda} = \dfrac{lf}{u_\text{p}} = \dfrac{(1\times10^{-3}\text{ m})\times(100\times10^{9}\text{ Hz})}{3\times10^{8}\text{ m/s}} = 0.33$ (nonnegligible).

Problem 2.2 Calculate the line parameters R', L', G', and C' for a coaxial line with an inner conductor diameter of 0.5 cm and an outer conductor diameter of 1 cm, filled with an insulating material where $\mu = \mu_0$, $\varepsilon_\text{r} = 4.5$, and $\sigma = 10^{-3}$ S/m. The conductors are made of copper with $\mu_\text{c} = \mu_0$ and $\sigma_\text{c} = 5.8\times10^{7}$ S/m. The operating frequency is 1 GHz.

Solution: Given

$$\begin{aligned} a &= (0.5/2)\text{ cm} = 0.25\times10^{-2}\text{ m},\\ b &= (1.0/2)\text{ cm} = 0.50\times10^{-2}\text{ m}, \end{aligned}$$

combining Eqs. (2.5) and (2.6) gives

$$\begin{aligned} R' &= \frac{1}{2\pi}\sqrt{\frac{\pi f\mu_\text{c}}{\sigma_\text{c}}}\left(\frac{1}{a}+\frac{1}{b}\right)\\ &= \frac{1}{2\pi}\sqrt{\frac{\pi(10^{9}\text{ Hz})(4\pi\times10^{-7}\text{ H/m})}{5.8\times10^{7}\text{ S/m}}}\left(\frac{1}{0.25\times10^{-2}\text{ m}}+\frac{1}{0.50\times10^{-2}\text{ m}}\right)\\ &= 0.788\ \Omega/\text{m}. \end{aligned}$$

From Eq. (2.7),

$$L' = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) = \frac{4\pi \times 10^{-7} \text{ H/m}}{2\pi} \ln 2 = 139 \text{ nH/m}.$$

From Eq. (2.8),

$$G' = \frac{2\pi\sigma}{\ln(b/a)} = \frac{2\pi \times 10^{-3} \text{ S/m}}{\ln 2} = 9.1 \text{ mS/m}.$$

From Eq. (2.9),

$$C' = \frac{2\pi\varepsilon}{\ln(b/a)} = \frac{2\pi\varepsilon_r\varepsilon_0}{\ln(b/a)} = \frac{2\pi \times 4.5 \times (8.854 \times 10^{-12} \text{ F/m})}{\ln 2} = 362 \text{ pF/m}.$$

Problem 2.3 A 1-GHz parallel-plate transmission line consists of 1.2-cm-wide copper strips separated by a 0.15-cm-thick layer of polystyrene. Appendix B gives $\mu_c = \mu_0 = 4\pi \times 10^{-7}$ (H/m) and $\sigma_c = 5.8 \times 10^7$ (S/m) for copper, and $\varepsilon_r = 2.6$ for polystyrene. Use Table 2-1 to determine the line parameters of the transmission line. Assume $\mu = \mu_0$ and $\sigma \simeq 0$ for polystyrene.

Solution:

$$\begin{aligned}
R' &= \frac{2R_s}{w} = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \frac{2}{1.2 \times 10^{-2}} \left(\frac{\pi \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}\right)^{1/2} = 1.38 \quad (\Omega/\text{m}), \\
L' &= \frac{\mu d}{w} = \frac{4\pi \times 10^{-7} \times 1.5 \times 10^{-3}}{1.2 \times 10^{-2}} = 1.57 \times 10^{-7} \quad (\text{H/m}), \\
G' &= 0 \qquad \text{because } \sigma = 0, \\
C' &= \frac{\varepsilon w}{d} = \varepsilon_0\varepsilon_r \frac{w}{d} = \frac{10^{-9}}{36\pi} \times 2.6 \times \frac{1.2 \times 10^{-2}}{1.5 \times 10^{-3}} = 1.84 \times 10^{-10} \quad (\text{F/m}).
\end{aligned}$$

Problem 2.4 Show that the transmission line model shown in Fig. 2-37 (P2.4) yields the same telegrapher's equations given by Eqs. (2.14) and (2.16).

Solution: The voltage at the central upper node is the same whether it is calculated from the left port or the right port:

$$\begin{aligned}
v(z + \tfrac{1}{2}\Delta z, t) &= v(z,t) - \tfrac{1}{2}R'\Delta z \, i(z,t) - \tfrac{1}{2}L'\Delta z \frac{\partial}{\partial t} i(z,t) \\
&= v(z + \Delta z, t) + \tfrac{1}{2}R'\Delta z \, i(z + \Delta z, t) + \tfrac{1}{2}L'\Delta z \frac{\partial}{\partial t} i(z + \Delta z, t).
\end{aligned}$$

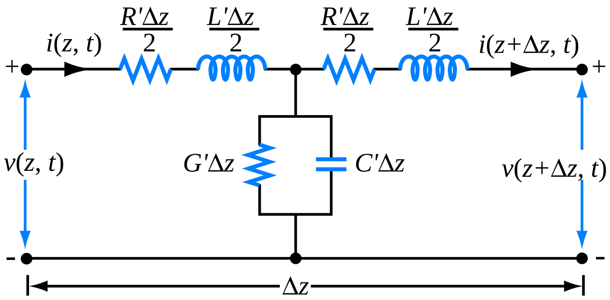


Figure P2.4: Transmission line model.

Recognizing that the current through the $G' \parallel C'$ branch is $i(z,t) - i(z+\Delta z,t)$ (from Kirchhoff's current law), we can conclude that

$$i(z,t) - i(z+\Delta z,t) = G'\Delta z\, v(z+\tfrac{1}{2}\Delta z,t) + C'\Delta z \frac{\partial}{\partial t} v(z+\tfrac{1}{2}\Delta z,t).$$

From both of these equations, the proof is completed by following the steps outlined in the text, ie. rearranging terms, dividing by Δz, and taking the limit as $\Delta z \to 0$.

Problem 2.5 Find α, β, u_p, and Z_0 for the coaxial line of Problem 2.2.

Solution: From Eq. (2.22),

$$\begin{aligned}\gamma &= \sqrt{(R' + j\omega L')(G' + j\omega C')} \\ &= \sqrt{(0.788\ \Omega/\text{m}) + j(2\pi \times 10^9\ \text{s}^{-1})(139 \times 10^{-9}\ \text{H/m})} \\ &\quad \times \sqrt{(9.1 \times 10^{-3}\ \text{S/m}) + j(2\pi \times 10^9\ \text{s}^{-1})(362 \times 10^{-12}\ \text{F/m})} \\ &= (109 \times 10^{-3} + j44.5)\ \text{m}^{-1}.\end{aligned}$$

Thus, from Eqs. (2.25a) and (2.25b), $\alpha = 0.109$ Np/m and $\beta = 44.5$ rad/m.

From Eq. (2.29),

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{(0.788\ \Omega/\text{m}) + j(2\pi \times 10^9\ \text{s}^{-1})(139 \times 10^{-9}\ \text{H/m})}{(9.1 \times 10^{-3}\ \text{S/m}) + j(2\pi \times 10^9\ \text{s}^{-1})(362 \times 10^{-12}\ \text{F/m})}}$$
$$= (19.6 + j0.030)\ \Omega.$$

From Eq. (2.33),

$$u_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^9}{44.5} = 1.41 \times 10^8\ \text{m/s}.$$

Section 2-5: The Lossless Line

Problem 2.6 In addition to not dissipating power, a lossless line has two important features: (1) it is dispertionless (μ_{p} is independent of frequency) and (2) its characteristic impedance Z_0 is purely real. Sometimes, it is not possible to design a transmission line such that $R' \ll \omega L'$ and $G' \ll \omega C'$, but it is possible to choose the dimensions of the line and its material properties so as to satisfy the condition

$$R'C' = L'G' \quad \text{(distortionless line).}$$

Such a line is called a *distortionless* line because despite the fact that it is not lossless, it does nonetheless possess the previously mentioned features of the loss line. Show that for a distortionless line,

$$\alpha = R'\sqrt{\frac{C'}{L'}} = \sqrt{R'G'}, \qquad \beta = \omega\sqrt{L'C'}, \qquad Z_0 = \sqrt{\frac{L'}{C'}}\,.$$

Solution: Using the distortionless condition in Eq. (2.22) gives

$$\begin{aligned}
\gamma = \alpha + j\beta &= \sqrt{(R' + j\omega L')(G' + j\omega C')} \\
&= \sqrt{L'C'}\sqrt{\left(\frac{R'}{L'} + j\omega\right)\left(\frac{G'}{C'} + j\omega\right)} \\
&= \sqrt{L'C'}\sqrt{\left(\frac{R'}{L'} + j\omega\right)\left(\frac{R'}{L'} + j\omega\right)} \\
&= \sqrt{L'C'}\left(\frac{R'}{L'} + j\omega\right) = R'\sqrt{\frac{C'}{L'}} + j\omega\sqrt{L'C'}.
\end{aligned}$$

Hence,

$$\alpha = \mathfrak{Re}(\gamma) = R'\sqrt{\frac{C'}{L'}}, \qquad \beta = \mathfrak{Im}(\gamma) = \omega\sqrt{L'C'}, \qquad u_{\mathrm{p}} = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}\,.$$

Similarly, using the distortionless condition in Eq. (2.29) gives

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{L'}{C'}}\sqrt{\frac{R'/L' + j\omega}{G'/C' + j\omega}} = \sqrt{\frac{L'}{C'}}\,.$$

Problem 2.7 For a distortionless line with $Z_0 = 50\ \Omega$, $\alpha = 20$ (mNp/m), $u_{\mathrm{p}} = 2.5 \times 10^8$ (m/s), find the line parameters and λ at 100 MHz.

Solution: The product of the expressions for α and Z_0 given in Problem 2.6 gives

$$R' = \alpha Z_0 = 20\times10^{-3}\times50 = 1 \quad (\Omega/\text{m}),$$

and taking the ratio of the expression for Z_0 to that for $u_\text{p} = \omega/\beta = 1/\sqrt{L'C'}$ gives

$$L' = \frac{Z_0}{u_\text{p}} = \frac{50}{2.5\times10^8} = 2\times10^{-7}\text{ (H/m)} = 200 \quad \text{(nH/m)}.$$

With L' known, we use the expression for Z_0 to find C':

$$C' = \frac{L'}{Z_0^2} = \frac{2\times10^{-7}}{(50)^2} = 8\times10^{-11}\text{ (F/m)} = 80 \quad \text{(pF/m)}.$$

The distortionless condition given in Problem 2.6 is then used to find G'.

$$G' = \frac{R'C'}{L'} = \frac{1\times80\times10^{-12}}{2\times10^{-7}} = 4\times10^{-4}\text{ (S/m)} = 400 \quad (\mu\text{S/m}),$$

and the wavelength is obtained by applying the relation

$$\lambda = \frac{\mu_\text{p}}{f} = \frac{2.5\times10^8}{100\times10^6} = 2.5\text{ m}.$$

Problem 2.8 Find α and Z_0 of a distortionless line whose $R' = 2$ Ω/m and $G' = 2\times10^{-4}$ S/m.

Solution: From the equations given in Problem 2.6,

$$\alpha = \sqrt{R'G'} = [2\times2\times10^{-4}]^{1/2} = 2\times10^{-2} \quad \text{(Np/m)},$$

$$Z_0 = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{R'}{G'}} = \left(\frac{2}{2\times10^{-4}}\right)^{1/2} = 100\ \Omega.$$

Problem 2.9 A transmission line operating at 125 MHz has $Z_0 = 40\ \Omega$, $\alpha = 0.02$ (Np/m), and $\beta = 0.75$ rad/m. Find the line parameters R', L', G', and C'.

Solution: Given an arbitrary transmission line, $f = 125$ MHz, $Z_0 = 40\ \Omega$, $\alpha = 0.02$ Np/m, and $\beta = 0.75$ rad/m. Since Z_0 is real and $\alpha \neq 0$, the line is distortionless. From Problem 2.6, $\beta = \omega\sqrt{L'C'}$ and $Z_0 = \sqrt{L'/C'}$, therefore,

$$L' = \frac{\beta Z_0}{\omega} = \frac{0.75\times40}{2\pi\times125\times10^6} = 38.2\text{ nH/m}.$$

Then, from $Z_0 = \sqrt{L'/C'}$,

$$C' = \frac{L'}{Z_0^2} = \frac{38.2 \text{ nH/m}}{40^2} = 23.9 \text{ pF/m}.$$

From $\alpha = \sqrt{R'G'}$ and $R'C' = L'G'$,

$$R' = \sqrt{R'G'}\sqrt{\frac{R'}{G'}} = \sqrt{R'G'}\sqrt{\frac{L'}{C'}} = \alpha Z_0 = 0.02 \text{ Np/m} \times 40\ \Omega = 0.6\ \Omega\text{/m}$$

and

$$G' = \frac{\alpha^2}{R'} = \frac{(0.02 \text{ Np/m})^2}{0.8\ \Omega\text{/m}} = 0.5 \text{ mS/m}.$$

Problem 2.10 Using a slotted line, the voltage on a lossless transmission line was found to have a maximum magnitude of 1.5 V and a minimum magnitude of 0.6 V. Find the magnitude of the load's reflection coefficient.

Solution: From the definition of the Standing Wave Ratio given by Eq. (2.59),

$$S = \frac{|\widetilde{V}|_{\max}}{|\widetilde{V}|_{\min}} = \frac{1.5}{0.6} = 2.5.$$

Solving for the magnitude of the reflection coefficient in terms of S, as in Example 2-4,

$$|\Gamma| = \frac{S-1}{S+1} = \frac{2.5-1}{2.5+1} = 0.43.$$

Problem 2.11 Polyethylene with $\varepsilon_r = 2.25$ is used as the insulating material in a lossless coaxial line with characteristic impedance of 50 Ω. The radius of the inner conductor is 1.2 mm.

(a) What is the radius of the outer conductor?

(b) What is the phase velocity of the line?

Solution: Given a lossless coaxial line, $Z_0 = 50\ \Omega$, $\varepsilon_r = 2.25$, $a = 1.2$ mm:

(a) From Table 2-2, $Z_0 = (60/\sqrt{\varepsilon_r})\ln(b/a)$ which can be rearranged to give

$$b = ae^{Z_0\sqrt{\varepsilon_r}/60} = (1.2 \text{ mm})e^{50\sqrt{2.25}/60} = 4.2 \text{ mm}.$$

(b) Also from Table 2-2,

$$u_\mathrm{p} = \frac{c}{\sqrt{\varepsilon_\mathrm{r}}} = \frac{3\times 10^8 \text{ m/s}}{\sqrt{2.25}} = 2.0\times 10^8 \text{ m/s}.$$

Problem 2.12 A 50-Ω lossless transmission line is terminated in a load with impedance $Z_\mathrm{L} = (30 - j50)$ Ω. The wavelength is 8 cm. Find:

(a) the reflection coefficient at the load,
(b) the standing-wave ratio on the line,
(c) the position of the voltage maximum nearest the load,
(d) the position of the current maximum nearest the load.

Solution:

(a) From Eq. (2.49a),

$$\Gamma = \frac{Z_\mathrm{L} - Z_0}{Z_\mathrm{L} + Z_0} = \frac{(30 - j50) - 50}{(30 - j50) + 50} = 0.57e^{-j79.8^\circ}.$$

(b) From Eq. (2.59),

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.57}{1 - 0.57} = 3.65.$$

(c) From Eq. (2.56)

$$\begin{aligned} l_\mathrm{max} &= \frac{\theta_\mathrm{r}\lambda}{4\pi} + \frac{n\lambda}{2} = \frac{-79.8^\circ \times 8 \text{ cm}}{4\pi}\frac{\pi \text{ rad}}{180^\circ} + \frac{n\times 8 \text{ cm}}{2} \\ &= -0.89 \text{ cm} + 4.0 \text{ cm} = 3.11 \text{ cm}. \end{aligned}$$

(d) A current maximum occurs at a voltage minimum, and from Eq. (2.58),

$$l_\mathrm{min} = l_\mathrm{max} - \lambda/4 = 3.11 \text{ cm} - 8 \text{ cm}/4 = 1.11 \text{ cm}.$$

Problem 2.13 On a 150-Ω lossless transmission line, the following observations were noted: distance of first voltage minimum from the load = 3 cm; distance of first voltage maximum from the load = 9 cm; $S = 3$. Find Z_L.

Solution: Distance between a minimum and an adjacent maximum = $\lambda/4$. Hence,

$$9 \text{ cm} - 3 \text{ cm} = 6 \text{ cm} = \lambda/4,$$

or $\lambda = 24$ cm. Accordingly, the first voltage minimum is at $\ell_{\text{min}} = 3 \text{ cm} = \frac{\lambda}{8}$. Application of Eq. (2.57) with $n = 0$ gives

$$\theta_r - 2 \times \frac{2\pi}{\lambda} \times \frac{\lambda}{8} = -\pi,$$

which gives $\theta_r = -\pi/2$.

$$|\Gamma| = \frac{S-1}{S+1} = \frac{3-1}{3+1} = \frac{2}{4} = 0.5.$$

Hence, $\Gamma = 0.5e^{-j\pi/2} = -j0.5$.

Finally,

$$Z_L = Z_0 \left[\frac{1+\Gamma}{1-\Gamma}\right] = 150 \left[\frac{1-j0.5}{1+j0.5}\right] = (90 - j120)\ \Omega.$$

Problem 2.14 Using a slotted line, the following results were obtained: distance of first minimum from the load = 4 cm; distance of second minimum from the load = 14 cm, voltage standing-wave ratio = 1.5. If the line is lossless and $Z_0 = 50\ \Omega$, find the load impedance.

Solution: Following Example 2.5: Given a lossless line with $Z_0 = 50\ \Omega$, $S = 1.5$, $l_{\text{min}(0)} = 4$ cm, $l_{\text{min}(1)} = 14$ cm. Then

$$l_{\text{min}(1)} - l_{\text{min}(0)} = \frac{\lambda}{2}$$

or

$$\lambda = 2 \times (l_{\text{min}(1)} - l_{\text{min}(0)}) = 20 \text{ cm}$$

and

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad/cycle}}{20 \text{ cm/cycle}} = 10\pi \text{ rad/m}.$$

From this we obtain

$$\begin{aligned}\theta_r = 2\beta l_{\text{min}(n)} - (2n+1)\pi \text{ rad} &= 2 \times 10\pi \text{ rad/m} \times 0.04 \text{ m} - \pi \text{ rad} \\ &= -0.2\pi \text{ rad} = -36.0^\circ.\end{aligned}$$

Also,

$$|\Gamma| = \frac{S-1}{S+1} = \frac{1.5-1}{1.5+1} = 0.2.$$

So

$$Z_L = Z_0 \left(\frac{1+\Gamma}{1-\Gamma} \right) = 50 \left(\frac{1+0.2e^{-j36.0^\circ}}{1-0.2e^{-j36.0^\circ}} \right) = (67.0 - j16.4)\ \Omega.$$

Problem 2.15 A load with impedance $Z_L = (25 - j50)\ \Omega$ is to be connected to a lossless transmission line with characteristic impedance Z_0, with Z_0 chosen such that the standing-wave ratio is the smallest possible. What should Z_0 be?

Solution: Since S is monotonic with $|\Gamma|$ (i.e., a plot of S *vs.* $|\Gamma|$ is always increasing), the value of Z_0 which gives the minimum possible S also gives the minimum possible $|\Gamma|$, and, for that matter, the minimum possible $|\Gamma|^2$. A necessary condition for a minimum is that its derivative be equal to zero:

$$\begin{aligned} 0 = \frac{\partial}{\partial Z_0}|\Gamma|^2 &= \frac{\partial}{\partial Z_0} \frac{|R_L + jX_L - Z_0|^2}{|R_L + jX_L + Z_0|^2} \\ &= \frac{\partial}{\partial Z_0} \frac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2} = \frac{4R_L(Z_0^2 - (R_L^2 + X_L^2))}{((R_L + Z_0)^2 + X_L^2)^2}. \end{aligned}$$

Therefore, $Z_0^2 = R_L^2 + X_L^2$ or

$$Z_0 = |Z_L| = \sqrt{(25^2 + (-50)^2)} = 55.9\ \Omega.$$

A mathematically precise solution will also demonstrate that this point is a minimum (by calculating the second derivative, for example). Since the endpoints of the range may be local minima or maxima without the derivative being zero there, the endpoints (namely $Z_0 = 0\ \Omega$ and $Z_0 = \infty\ \Omega$) should be checked also.

Problem 2.16 A 50-Ω lossless line terminated in a purely resistive load has a voltage standing wave ratio of 3. Find all possible values of Z_L.

Solution:

$$|\Gamma| = \frac{S-1}{S+1} = \frac{3-1}{3+1} = 0.5.$$

For a purely resistive load, $\theta_r = 0$ or π. For $\theta_r = 0$,

$$Z_L = Z_0 \left[\frac{1+\Gamma}{1-\Gamma} \right] = 50 \left[\frac{1+0.5}{1-0.5} \right] = 150\ \Omega.$$

For $\theta_r = \pi$, $\Gamma = -0.5$ and

$$Z_L = 50 \left[\frac{1-0.5}{1+0.5} \right] = 15\ \Omega.$$

Section 2-6: Input Impedance

Problem 2.17 At an operating frequency of 300 MHz, a lossless 50-Ω air-spaced transmission line 2.5 m in length is terminated with an impedance $Z_L = (40 + j20)\ \Omega$. Find the input impedance.

Solution: Given a lossless transmission line, $Z_0 = 50\ \Omega$, $f = 300$ MHz, $l = 2.5$ m, and $Z_L = (40 + j20)\ \Omega$. Since the line is air filled, $u_p = c$ and therefore, from Eq. (2.38),

$$\beta = \frac{\omega}{u_p} = \frac{2\pi \times 300 \times 10^6}{3 \times 10^8} = 2\pi \text{ rad/m}.$$

Since the line is lossless, Eq. (2.69) is valid:

$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \right) = 50 \left(\frac{(40 + j20) + j50 \tan(2\pi \text{ rad/m} \times 2.5 \text{ m})}{50 + j(40 + j20) \tan(2\pi \text{ rad/m} \times 2.5 \text{ m})} \right)$$
$$= 50 \left(\frac{(40 + j20) + j50 \times 0}{50 + j(40 + j20) \times 0} \right) = (40 + j20)\ \Omega.$$

Problem 2.18 A lossless transmission line of electrical length $l = 0.35\lambda$ is terminated in a load impedance as shown in Fig. 2-38 (P2.18). Find Γ, S, and Z_{in}.

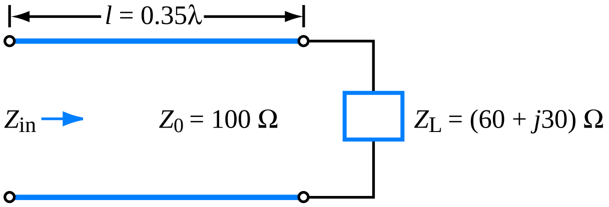


Figure P2.18: Loaded transmission line.

Solution: From Eq. (2.49a),

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(60 + j30) - 100}{(60 + j30) + 100} = 0.307 e^{j132.5^\circ}.$$

From Eq. (2.59),

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.307}{1 - 0.307} = 1.89.$$

From Eq. (2.63)

$$Z_{\text{in}} = Z_0 \left(\frac{Z_{\text{L}} + jZ_0 \tan\beta l}{Z_0 + jZ_{\text{L}} \tan\beta l} \right)$$
$$= 100 \left(\frac{(60 + j30) + j100 \tan\left(\frac{2\pi \text{ rad}}{\lambda} 0.35\lambda\right)}{100 + j(60 + j30) \tan\left(\frac{2\pi \text{ rad}}{\lambda} 0.35\lambda\right)} \right) = (64.8 - j38.3)\ \Omega.$$

Problem 2.19 Show that the input impedance of a quarter-wavelength long lossless line terminated in a short circuit appears as an open circuit.

Solution:

$$Z_{\text{in}} = Z_0 \left(\frac{Z_{\text{L}} + jZ_0 \tan\beta l}{Z_0 + jZ_{\text{L}} \tan\beta l} \right).$$

For $l = \frac{\lambda}{4}$, $\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$. With $Z_{\text{L}} = 0$, we have

$$Z_{\text{in}} = Z_0 \left(\frac{jZ_0 \tan\pi/2}{Z_0} \right) = j\infty \quad \text{(open circuit).}$$

Problem 2.20 Show that at the position where the magnitude of the voltage on the line is a maximum the input impedance is purely real.

Solution: From Eq. (2.56), $l_{\max} = (\theta_{\text{r}} + 2n\pi)/2\beta$, so from Eq. (2.61), using polar representation for Γ,

$$Z_{\text{in}}(-l_{\max}) = Z_0 \left(\frac{1 + |\Gamma| e^{j\theta_{\text{r}}} e^{-j2\beta l_{\max}}}{1 - |\Gamma| e^{j\theta_{\text{r}}} e^{-j2\beta l_{\max}}} \right)$$
$$= Z_0 \left(\frac{1 + |\Gamma| e^{j\theta_{\text{r}}} e^{-j(\theta_{\text{r}} + 2n\pi)}}{1 - |\Gamma| e^{j\theta_{\text{r}}} e^{-j(\theta_{\text{r}} + 2n\pi)}} \right) = Z_0 \left(\frac{1 + |\Gamma|}{1 - |\Gamma|} \right),$$

which is real, provided Z_0 is real.

Problem 2.21 A voltage generator with $v_{\text{g}}(t) = 5\cos(2\pi \times 10^9 t)$ V and internal impedance $Z_{\text{g}} = 50\ \Omega$ is connected to a 50-Ω lossless air-spaced transmission line. The line length is 5 cm and it is terminated in a load with impedance $Z_{\text{L}} = (100 - j100)\ \Omega$. Find

(a) Γ at the load.

(b) Z_{in} at the input to the transmission line.

(c) the input voltage $\widetilde{V}_{\text{i}}$ and input current $\tilde{I}_{\text{i}}$.

Solution:

(a) From Eq. (2.49a),

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(100 - j100) - 50}{(100 - j100) + 50} = 0.62e^{-j29.7^\circ}.$$

(b) All formulae for Z_{in} require knowledge of $\beta = \omega/u_p$. Since the line is an air line, $u_p = c$, and from the expression for $v_g(t)$ we conclude $\omega = 2\pi \times 10^9$ rad/s. Therefore

$$\beta = \frac{2\pi \times 10^9 \text{ rad/s}}{3 \times 10^8 \text{ m/s}} = \frac{20\pi}{3} \text{ rad/m}.$$

Then, using Eq. (2.63),

$$\begin{aligned} Z_{in} &= Z_0 \left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right) \\ &= 50 \left(\frac{(100 - j100) + j50 \tan\left(\frac{20\pi}{3} \text{ rad/m} \times 5 \text{ cm}\right)}{50 + j(100 - j100) \tan\left(\frac{20\pi}{3} \text{ rad/m} \times 5 \text{ cm}\right)} \right) \\ &= 50 \left(\frac{(100 - j100) + j50 \tan\left(\frac{\pi}{3} \text{ rad}\right)}{50 + j(100 - j100) \tan\left(\frac{\pi}{3} \text{ rad}\right)} \right) = (12.5 - j12.7)\ \Omega. \end{aligned}$$

An alternative solution to this part involves the solution to part (a) and Eq. (2.61).

(c) In phasor domain, $\widetilde{V}_g = 5 \text{ V } e^{j0^\circ}$. From Eq. (2.64),

$$\widetilde{V}_i = \frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}} = \frac{5 \times (12.5 - j12.7)}{50 + (12.5 - j12.7)} = 1.40e^{-j34.0^\circ} \quad \text{(V)},$$

and also from Eq. (2.64),

$$\widetilde{I}_i = \frac{\widetilde{V}_i}{Z_{in}} = \frac{1.4e^{-j34.0^\circ}}{(12.5 - j12.7)} = 78.4e^{j11.5^\circ} \quad \text{(mA)}.$$

Problem 2.22 A 6-m section of 150-Ω lossless line is driven by a source with

$$v_g(t) = 5\cos(8\pi \times 10^7 t - 30^\circ) \quad \text{(V)}$$

and $Z_g = 150\ \Omega$. If the line, which has a relative permittivity $\varepsilon_r = 2.25$, is terminated in a load $Z_L = (150 - j50)\ \Omega$, find

(a) λ on the line,

(b) the reflection coefficient at the load,

(c) the input impedance,

(d) the input voltage $\widetilde{V}_{\mathrm{i}}$,

(e) the time-domain input voltage $v_{\mathrm{i}}(t)$.

Solution:

$$v_{\mathrm{g}}(t) = 5\cos(8\pi \times 10^7 t - 30^\circ) \;\; \mathrm{V},$$
$$\widetilde{V}_{\mathrm{g}} = 5e^{-j30^\circ} \;\; \mathrm{V}.$$

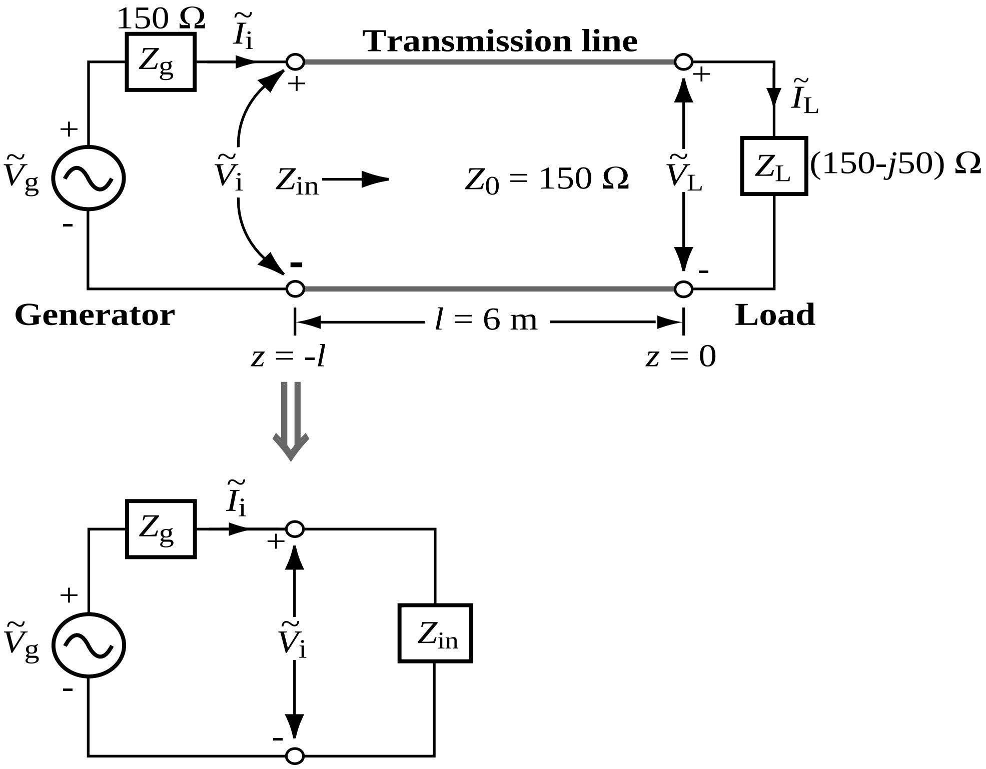


Figure P2.22: Circuit for Problem 2.22.

(a)

$$u_{\mathrm{p}} = \frac{c}{\sqrt{\varepsilon_{\mathrm{r}}}} = \frac{3\times10^8}{\sqrt{2.25}} = 2\times10^8 \quad \text{(m/s)},$$
$$\lambda = \frac{u_{\mathrm{p}}}{f} = \frac{2\pi u_{\mathrm{p}}}{\omega} = \frac{2\pi\times2\times10^8}{8\pi\times10^7} = 5 \;\; \mathrm{m},$$
$$\beta = \frac{\omega}{u_{\mathrm{p}}} = \frac{8\pi\times10^7}{2\times10^8} = 0.4\pi \quad \text{(rad/m)},$$
$$\beta l = 0.4\pi \times 6 = 2.4\pi \quad \text{(rad)}.$$

Since this exceeds 2π (rad), we can subtract 2π, which leaves a remainder $\beta l = 0.4\pi$ (rad).

(b) $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{150 - j50 - 150}{150 - j50 + 150} = \dfrac{-j50}{300 - j50} = 0.16e^{-j80.54^\circ}$.

(c)

$$\begin{aligned} Z_{\text{in}} &= Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right] \\ &= 150\left[\frac{(150 - j50) + j150\tan(0.4\pi)}{150 + j(150 - j50)\tan(0.4\pi)}\right] = (115.70 + j27.42)\ \Omega. \end{aligned}$$

(d)

$$\begin{aligned} \widetilde{V}_i = \frac{\widetilde{V}_g Z_{\text{in}}}{Z_g + Z_{\text{in}}} &= \frac{5e^{-j30^\circ}(115.7 + j27.42)}{150 + 115.7 + j27.42} \\ &= 5e^{-j30^\circ}\left(\frac{115.7 + j27.42}{265.7 + j27.42}\right) \\ &= 5e^{-j30^\circ} \times 0.44e^{j7.44^\circ} = 2.2e^{-j22.56^\circ} \quad \text{(V)}. \end{aligned}$$

(e)

$$v_i(t) = \mathfrak{Re}[\widetilde{V}_i e^{j\omega t}] = \mathfrak{Re}[2.2e^{-j22.56^\circ}e^{j\omega t}] = 2.2\cos(8\pi \times 10^7 t - 22.56^\circ)\ \text{V}.$$

Problem 2.23 Two half-wave dipole antennas, each with impedance of 75 Ω, are connected in parallel through a pair of transmission lines, and the combination is connected to a feed transmission line, as shown in Fig. 2.39 (P2.23(a)). All lines are 50 Ω and lossless.

(a) Calculate Z_{in_1}, the input impedance of the antenna-terminated line, at the parallel juncture.

(b) Combine Z_{in_1} and Z_{in_2} in parallel to obtain Z'_L, the effective load impedance of the feedline.

(c) Calculate Z_{in} of the feedline.

Solution:

(a)

$$\begin{aligned} Z_{\text{in}_1} &= Z_0 \left[\frac{Z_{L_1} + jZ_0 \tan\beta l_1}{Z_0 + jZ_{L_1} \tan\beta l_1}\right] \\ &= 50\left\{\frac{75 + j50\tan[(2\pi/\lambda)(0.2\lambda)]}{50 + j75\tan[(2\pi/\lambda)(0.2\lambda)]}\right\} = (35.20 - j8.62)\ \Omega. \end{aligned}$$

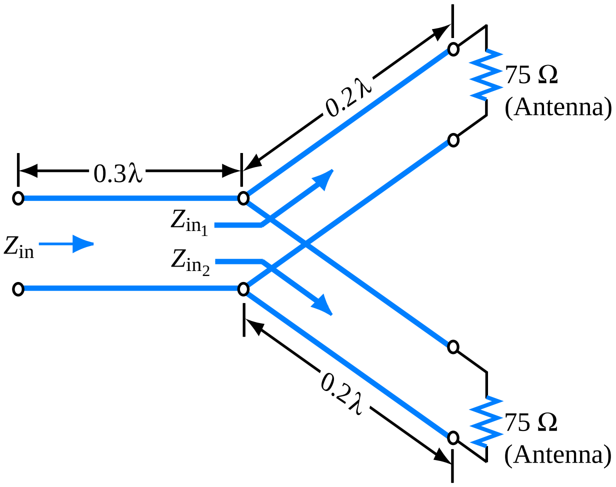


Figure P2.23: (a) Circuit for Problem 2.23.

(b)

$$Z'_L = \frac{Z_{in_1} Z_{in_2}}{Z_{in_1} + Z_{in_2}} = \frac{(35.20 - j8.62)^2}{2(35.20 - j8.62)} = (17.60 - j4.31)\ \Omega.$$

(c)

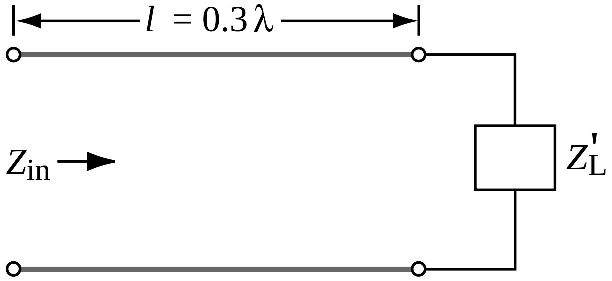


Figure P2.23: (b) Equivalent circuit.

$$Z_{in} = 50 \left\{ \frac{(17.60 - j4.31) + j50 \tan[(2\pi/\lambda)(0.3\lambda)]}{50 + j(17.60 - j4.31) \tan[(2\pi/\lambda)(0.3\lambda)]} \right\} = (107.57 - j56.7)\ \Omega.$$

Section 2-7: Special Cases

Problem 2.24 At an operating frequency of 300 MHz, it is desired to use a section of a lossless 50-Ω transmission line terminated in a short circuit to construct an equivalent load with reactance $X = 40\ \Omega$. If the phase velocity of the line is $0.75c$, what is the shortest possible line length that would exhibit the desired reactance at its input?

Solution:

$$\beta = \omega/u_{\rm p} = \frac{(2\pi \text{ rad/cycle}) \times (300 \times 10^6 \text{ cycle/s})}{0.75 \times (3 \times 10^8 \text{ m/s})} = 8.38 \text{ rad/m}.$$

On a lossless short-circuited transmission line, the input impedance is always purely imaginary; i.e., $Z_{\rm in}^{\rm sc} = jX_{\rm in}^{\rm sc}$. Solving Eq. (2.68) for the line length,

$$l = \frac{1}{\beta}\tan^{-1}\left(\frac{X_{\rm in}^{\rm sc}}{Z_0}\right) = \frac{1}{8.38 \text{ rad/m}}\tan^{-1}\left(\frac{40\ \Omega}{50\ \Omega}\right) = \frac{(0.675 + n\pi) \text{ rad}}{8.38 \text{ rad/m}},$$

for which the smallest positive solution is 8.05 cm (with $n = 0$).

Problem 2.25 A lossless transmission line is terminated in a short circuit. How long (in wavelengths) should the line be in order for it to appear as an open circuit at its input terminals?

Solution: From Eq. (2.68), $Z_{\rm in}^{\rm sc} = jZ_0 \tan\beta l$. If $\beta l = (\pi/2 + n\pi)$, then $Z_{\rm in}^{\rm sc} = j\infty\ (\Omega)$. Hence,

$$l = \frac{\lambda}{2\pi}\left(\frac{\pi}{2} + n\pi\right) = \frac{\lambda}{4} + \frac{n\lambda}{2}.$$

This is evident from Figure 2.15(d).

Problem 2.26 The input impedance of a 31-cm-long lossless transmission line of unknown characteristic impedance was measured at 1 MHz. With the line terminated in a short circuit, the measurement yielded an input impedance equivalent to an inductor with inductance of 0.064 μH, and when the line was open circuited, the measurement yielded an input impedance equivalent to a capacitor with capacitance of 40 pF. Find Z_0 of the line, the phase velocity, and the relative permittivity of the insulating material.

Solution: Now $\omega = 2\pi f = 6.28 \times 10^6$ rad/s, so

$$Z_{\rm in}^{\rm sc} = j\omega L = j2\pi \times 10^6 \times 0.064 \times 10^{-6} = j0.4\ \Omega$$

and $Z_{\text{in}}^{\text{oc}} = 1/j\omega C = 1/(j2\pi \times 10^6 \times 40 \times 10^{-12}) = -j4000\ \Omega$.

From Eq. (2.74), $Z_0 = \sqrt{Z_{\text{in}}^{\text{sc}} Z_{\text{in}}^{\text{oc}}} = \sqrt{(j0.4\ \Omega)(-j4000\ \Omega)} = 40\ \Omega$. Using Eq. (2.75),

$$\begin{aligned} u_{\text{p}} = \frac{\omega}{\beta} &= \frac{\omega l}{\tan^{-1}\sqrt{-Z_{\text{in}}^{\text{sc}}/Z_{\text{in}}^{\text{oc}}}} \\ &= \frac{6.28 \times 10^6 \times 0.31}{\tan^{-1}\left(\pm\sqrt{-j0.4/(-j4000)}\right)} = \frac{1.95 \times 10^6}{(\pm 0.01 + n\pi)} \text{ m/s}, \end{aligned}$$

where $n \geq 0$ for the plus sign and $n \geq 1$ for the minus sign. For $n = 0$, $u_{\text{p}} = 1.94 \times 10^8$ m/s $= 0.65c$ and $\varepsilon_{\text{r}} = (c/u_{\text{p}})^2 = 1/0.65^2 = 2.4$. For other values of n, u_{p} is very slow and ε_{r} is unreasonably high.

Problem 2.27 A 75-Ω resistive load is preceded by a $\lambda/4$ section of a 50-Ω lossless line, which itself is preceded by another $\lambda/4$ section of a 100-Ω line. What is the input impedance?

Solution: The input impedance of the $\lambda/4$ section of line closest to the load is found from Eq. (2.77):

$$Z_{\text{in}} = \frac{Z_0^2}{Z_{\text{L}}} = \frac{50^2}{75} = 33.33\ \Omega.$$

The input impedance of the line section closest to the load can be considered as the load impedance of the next section of the line. By reapplying Eq. (2.77), the next section of $\lambda/4$ line is taken into account:

$$Z_{\text{in}} = \frac{Z_0^2}{Z_{\text{L}}} = \frac{100^2}{33.33} = 300\ \Omega.$$

Problem 2.28 A 100-MHz FM broadcast station uses a 300-Ω transmission line between the transmitter and a tower-mounted half-wave dipole antenna. The antenna impedance is 73 Ω. You are asked to design a quarter-wave transformer to match the antenna to the line.

(a) Determine the electrical length and characteristic impedance of the quarter-wave section.

(b) If the quarter-wave section is a two-wire line with $d = 2.5$ cm, and the spacing between the wires is made of polystyrene with $\varepsilon_{\text{r}} = 2.6$, determine the physical length of the quarter-wave section and the radius of the two wire conductors.

Solution:

(a) For a match condition, the input impedance of a load must match that of the transmission line attached to the generator. A line of electrical length $\lambda/4$ can be used. From Eq. (2.77), the impedance of such a line should be

$$Z_0 = \sqrt{Z_{\rm in} Z_{\rm L}} = \sqrt{300 \times 73} = 148\ \Omega.$$

(b)

$$\frac{\lambda}{4} = \frac{u_{\rm p}}{4f} = \frac{c}{4\sqrt{\varepsilon_{\rm r}}\,f} = \frac{3 \times 10^8}{4\sqrt{2.6} \times 100 \times 10^6} = 0.465\ \text{m},$$

and, from Table 2-2,

$$Z_0 = \frac{120}{\sqrt{\varepsilon}} \ln\left[\left(\frac{d}{2a}\right) + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right]\ \Omega.$$

Hence,

$$\ln\left[\left(\frac{d}{2a}\right) + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right] = \frac{148\sqrt{2.6}}{120} = 1.99,$$

which leads to

$$\left(\frac{d}{2a}\right) + \sqrt{\left(\frac{d}{2a}\right)^2 - 1} = 7.31,$$

and whose solution is $a = d/7.44 = 25\ \text{cm}/7.44 = 3.36$ mm.

Problem 2.29 A 50-MHz generator with $Z_{\rm g} = 50\ \Omega$ is connected to a load $Z_{\rm L} = (50 - j25)\ \Omega$. The time-average power transferred from the generator into the load is maximum when $Z_{\rm g} = Z_{\rm L}^*$, where $Z_{\rm L}^*$ is the complex conjugate of $Z_{\rm L}$. To achieve this condition without changing $Z_{\rm g}$, the effective load impedance can be modified by adding an open-circuited line in series with $Z_{\rm L}$, as shown in Fig. 2-40 (P2.29). If the line's $Z_0 = 100\ \Omega$, determine the shortest length of line (in wavelengths) necessary for satisfying the maximum-power-transfer condition.

Solution: Since the real part of $Z_{\rm L}$ is equal to $Z_{\rm g}$, our task is to find l such that the input impedance of the line is $Z_{\rm in} = +j25\ \Omega$, thereby cancelling the imaginary part of $Z_{\rm L}$ (once $Z_{\rm L}$ and the input impedance the line are added in series). Hence, using Eq. (2.73),

$$-j100 \cot\beta l = j25,$$

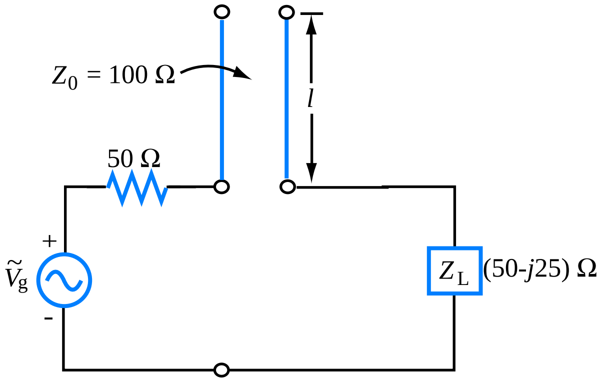


Figure P2.29: Transmission-line arrangement for Problem 2.29.

or

$$\cot \beta l = -\frac{25}{100} = -0.25,$$

which leads to

$$\beta l = -1.326 \text{ or } 1.816.$$

Since l cannot be negative, the first solution is discarded. The second solution leads to

$$l = \frac{1.816}{\beta} = \frac{1.816}{(2\pi/\lambda)} = 0.29\lambda.$$

Problem 2.30 A 50-Ω lossless line of length $l = 0.375\lambda$ connects a 300-MHz generator with $\widetilde{V}_g = 300$ V and $Z_g = 50$ Ω to a load Z_L. Determine the time-domain current through the load for:

(a) $Z_L = (50 - j50)$ Ω,

(b) $Z_L = 50$ Ω,

(c) $Z_L = 0$ (short circuit).

Solution:

(a) $Z_L = (50 - j50)$ Ω, $\beta l = \frac{2\pi}{\lambda} \times 0.375\lambda = 2.36 \text{ (rad)} = 135°$.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j50 - 50}{50 - j50 + 50} = \frac{-j50}{100 - j50} = 0.45\, e^{-j63.43°}.$$

Application of Eq. (2.63) gives:

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}\right] = 50 \left[\frac{(50 - j50) + j50 \tan 135°}{50 + j(50 - j50) \tan 135°}\right] = (100 + j50) \ \Omega.$$

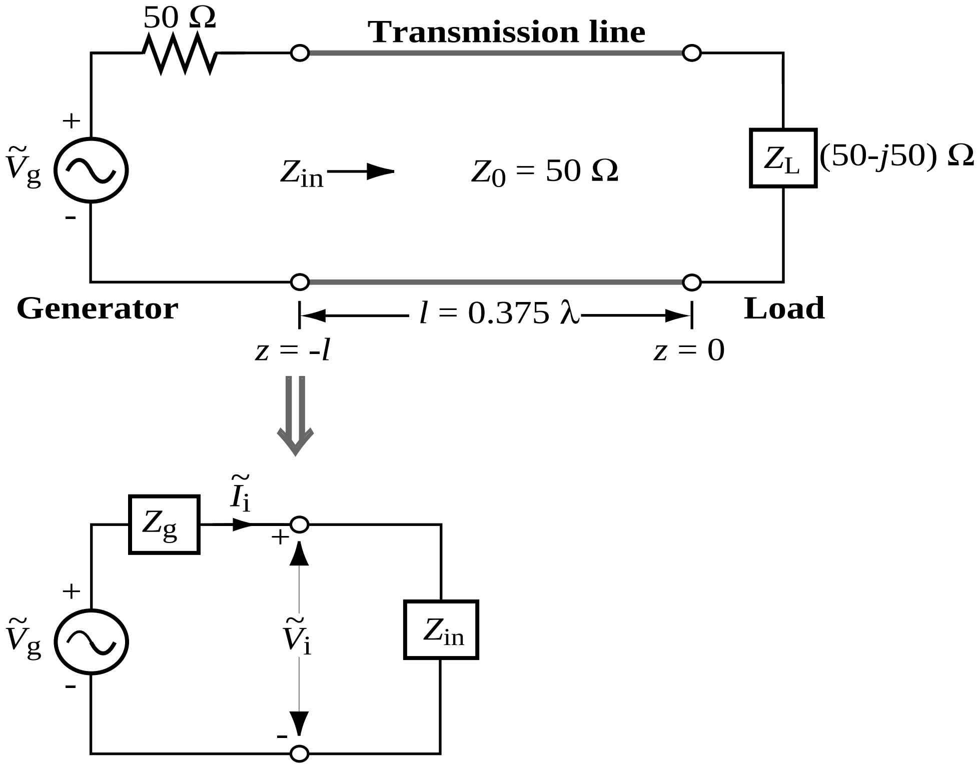


Figure P2.30: Circuit for Problem 2.30(a).

Using Eq. (2.66) gives

$$
\begin{aligned}
V_0^+ &= \left(\frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) \\
&= \frac{300(100 + j50)}{50 + (100 + j50)}\left(\frac{1}{e^{j135^\circ} + 0.45\,e^{-j63.43^\circ}\,e^{-j135^\circ}}\right) \\
&= 150\,e^{-j135^\circ} \quad \text{(V)}, \\
\widetilde{I}_L &= \frac{V_0^+}{Z_0}(1 - \Gamma) = \frac{150\,e^{-j135^\circ}}{50}(1 - 0.45\,e^{-j63.43^\circ}) = 2.68\,e^{-j108.44^\circ} \quad \text{(A)}, \\
i_L(t) &= \mathfrak{Re}[\widetilde{I}_L e^{j\omega t}] \\
&= \mathfrak{Re}[2.68\,e^{-j108.44^\circ}\,e^{j6\pi\times 10^8 t}] \\
&= 2.68\cos(6\pi \times 10^8 t - 108.44^\circ) \quad \text{(A)}.
\end{aligned}
$$

(b)

$$\begin{aligned}
Z_L &= 50\ \Omega,\\
\Gamma &= 0,\\
Z_{in} &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad Z_0 = 50\ \Omega,\\
V_0^+ &= \frac{300\times 50}{50+50}\left(\frac{1}{e^{j135^\circ}+0}\right) = 150\,e^{-j135^\circ} \quad \text{(V)},\\
\widetilde{I}_L &= \frac{V_0^+}{Z_0} = \frac{150}{50}\,e^{-j135^\circ} = 3\,e^{-j135^\circ} \quad \text{(A)},\\
i_L(t) &= \mathfrak{Re}[3\,e^{-j135^\circ}e^{j6\pi\times10^8t}] = 3\cos(6\pi\times10^8t-135^\circ) \quad \text{(A)}.
\end{aligned}$$

(c)

$$\begin{aligned}
Z_L &= 0,\\
\Gamma &= -1,\\
Z_{in} &= Z_0\left(\frac{0+jZ_0\tan 135^\circ}{Z_0+0}\right) = jZ_0\tan 135^\circ = -j50 \quad (\Omega),\\
V_0^+ &= \frac{300(-j50)}{50-j50}\left(\frac{1}{e^{j135^\circ}-e^{-j135^\circ}}\right) = 150\,e^{-j135^\circ} \quad \text{(V)},\\
\widetilde{I}_L &= \frac{V_0^+}{Z_0}[1-\Gamma] = \frac{150\,e^{-j135^\circ}}{50}[1+1] = 6e^{-j135^\circ} \quad \text{(A)},\\
i_L(t) &= 6\cos(6\pi\times10^8t-135^\circ) \quad \text{(A)}.
\end{aligned}$$

Section 2-8: Power Flow on Lossless Line

Problem 2.31 A generator with $\widetilde{V}_g = 300$ V and $Z_g = 50\ \Omega$ is connected to a load $Z_L = 75\ \Omega$ through a 50-Ω lossless line of length $l = 0.15\lambda$.

- **(a)** Compute Z_{in}, the input impedance of the line at the generator end.
- **(b)** Compute $\widetilde{I}_i$ and $\widetilde{V}_i$.
- **(c)** Compute the time-average power delivered to the line, $P_{in} = \frac{1}{2}\mathfrak{Re}[\widetilde{V}_i\widetilde{I}_i^*]$.
- **(d)** Compute $\widetilde{V}_L$, $\widetilde{I}_L$, and the time-average power delivered to the load, $P_L = \frac{1}{2}\mathfrak{Re}[\widetilde{V}_L\widetilde{I}_L^*]$. How does P_{in} compare to P_L? Explain.
- **(e)** Compute the time average power delivered by the generator, P_g, and the time average power dissipated in Z_g. Is conservation of power satisfied?

Solution:

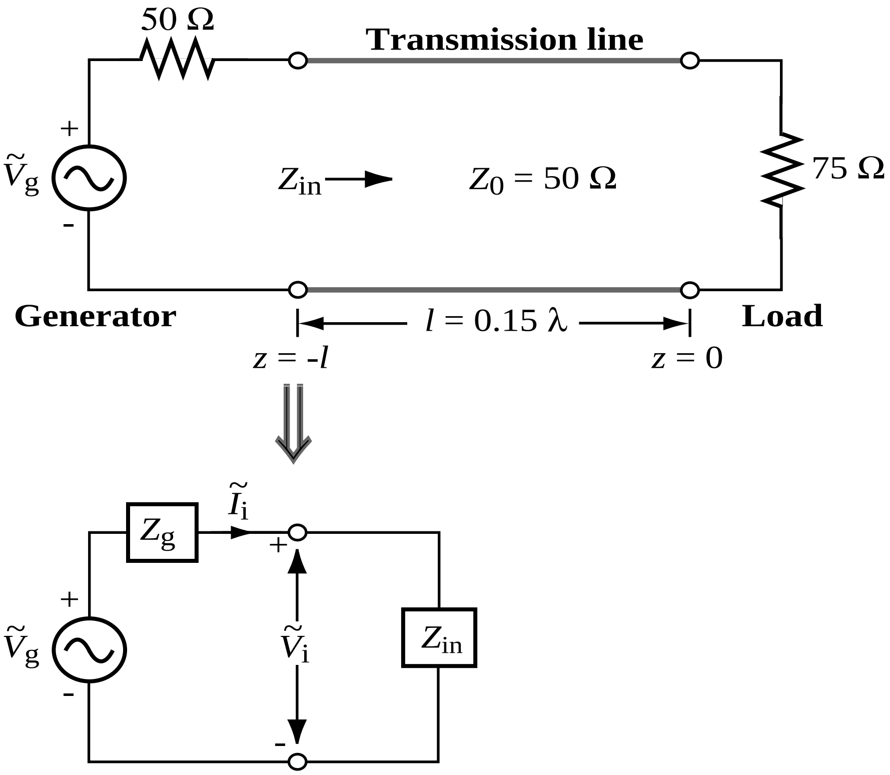


Figure P2.31: Circuit for Problem 2.31.

(a)

$$\beta l = \frac{2\pi}{\lambda} \times 0.15\lambda = 54^\circ,$$

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right] = 50 \left[\frac{75 + j50\tan 54^\circ}{50 + j75\tan 54^\circ}\right] = (41.25 - j16.35)\ \Omega.$$

(b)

$$\widetilde{I}_i = \frac{\widetilde{V}_g}{Z_g + Z_{in}} = \frac{300}{50 + (41.25 - j16.35)} = 3.24e^{j10.16^\circ} \quad \text{(A)},$$

$$\widetilde{V}_i = \widetilde{I}_i Z_{in} = 3.24e^{j10.16^\circ}(41.25 - j16.35) = 143.6e^{-j11.46^\circ} \quad \text{(V)}.$$

(c)

$$
\begin{aligned}
P_{\text{in}} = \frac{1}{2}\mathfrak{Re}[\widetilde{V}_{\text{i}}\widetilde{I}_{\text{i}}^{*}] &= \frac{1}{2}\mathfrak{Re}[143.6\,e^{-j11.46^\circ} \times 3.24\,e^{-j10.16^\circ}] \\
&= \frac{143.6 \times 3.24}{2}\cos(21.62^\circ) = 216 \quad \text{(W)}.
\end{aligned}
$$

(d)

$$
\begin{aligned}
\Gamma &= \frac{Z_{\text{L}} - Z_0}{Z_{\text{L}} + Z_0} = \frac{75 - 50}{75 + 50} = 0.2, \\
V_0^{+} &= \widetilde{V}_{\text{i}}\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) = \frac{143.6\,e^{-j11.46^\circ}}{e^{j54^\circ} + 0.2\,e^{-j54^\circ}} = 150e^{-j54^\circ} \quad \text{(V)}, \\
\widetilde{V}_{\text{L}} &= V_0^{+}(1 + \Gamma) = 150e^{-j54^\circ}(1 + 0.2) = 180e^{-j54^\circ} \quad \text{(V)}, \\
\widetilde{I}_{\text{L}} &= \frac{V_0^{+}}{Z_0}(1 - \Gamma) = \frac{150e^{-j54^\circ}}{50}(1 - 0.2) = 2.4\,e^{-j54^\circ} \quad \text{(A)}, \\
P_{\text{L}} &= \frac{1}{2}\mathfrak{Re}[\widetilde{V}_{\text{L}}\widetilde{I}_{\text{L}}^{*}] = \frac{1}{2}\mathfrak{Re}[180e^{-j54^\circ} \times 2.4\,e^{j54^\circ}] = 216 \quad \text{(W)}.
\end{aligned}
$$

$P_{\text{L}} = P_{\text{in}}$, which is as expected because the line is lossless; power input to the line ends up in the load.

(e)

Power delivered by generator:

$$
P_{\text{g}} = \frac{1}{2}\mathfrak{Re}[\widetilde{V}_{\text{g}}\widetilde{I}_{\text{i}}] = \frac{1}{2}\mathfrak{Re}[300 \times 3.24\,e^{j10.16^\circ}] = 486\cos(10.16^\circ) = 478.4 \quad \text{(W)}.
$$

Power dissipated in Z_{g}:

$$
P_{Z_{\text{g}}} = \frac{1}{2}\mathfrak{Re}[\widetilde{I}_{\text{i}}\widetilde{V}_{Z_{\text{g}}}] = \frac{1}{2}\mathfrak{Re}[\widetilde{I}_{\text{i}}\widetilde{I}_{\text{i}}^{*}Z_{\text{g}}] = \frac{1}{2}|\widetilde{I}_{\text{i}}|^2 Z_{\text{g}} = \frac{1}{2}(3.24)^2 \times 50 = 262.4 \quad \text{(W)}.
$$

Note 1: $P_{\text{g}} = P_{Z_{\text{g}}} + P_{\text{in}} = 478.4$ W.

Problem 2.32 If the two-antenna configuration shown in Fig. 2-41 (P2.32) is connected to a generator with $\widetilde{V}_{\text{g}} = 250$ V and $Z_{\text{g}} = 50\ \Omega$, how much average power is delivered to each antenna?

Solution: Since line 2 is $\lambda/2$ in length, the input impedance is the same as $Z_{\text{L}_1} = 75\ \Omega$. The same is true for line 3. At junction C–D, we now have two 75-Ω impedances in parallel, whose combination is $75/2 = 37.5\ \Omega$. Line 1 is $\lambda/2$ long. Hence at A–C, input impedance of line 1 is 37.5 Ω, and

$$
\widetilde{I}_{\text{i}} = \frac{\widetilde{V}_{\text{g}}}{Z_{\text{g}} + Z_{\text{in}}} = \frac{250}{50 + 37.5} = 2.86 \quad \text{(A)},
$$

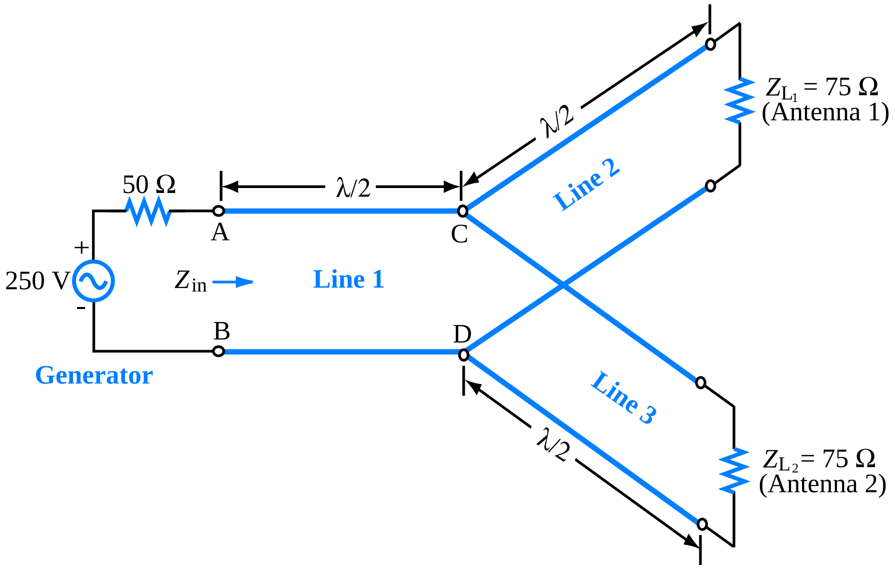


Figure P2.32: Antenna configuration for Problem 2.32.

$$P_{\text{in}} = \frac{1}{2}\mathfrak{Re}[\widetilde{I}_{\text{i}}\widetilde{V}_{\text{i}}^*] = \frac{1}{2}\mathfrak{Re}[\widetilde{I}_{\text{i}}\widetilde{I}_{\text{i}}^*\widetilde{Z}_{\text{in}}^*] = \frac{(2.86)^2 \times 37.5}{2} = 153.37 \quad \text{(W)}.$$

This is divided equally between the two antennas. Hence, each antenna receives $\frac{153.37}{2} = 76.68$ (W).

Problem 2.33 For the circuit shown in Fig. 2-42 (P2.33), calculate the average incident power, the average reflected power, and the average power transmitted into the infinite 100-Ω line. The $\lambda/2$ line is lossless and the infinitely long line is slightly lossy. (Hint: The input impedance of an infinitely long line is equal to its characteristic impedance so long as $\alpha \neq 0$.)

Solution: Considering the semi-infinite transmission line as equivalent to a load (since all power sent down the line is lost to the rest of the circuit), $Z_{\text{L}} = Z_1 = 100$ Ω. Since the feed line is $\lambda/2$ in length, Eq. (2.76) gives $Z_{\text{in}} = Z_{\text{L}} = 100$ Ω and $\beta l = (2\pi/\lambda)(\lambda/2) = \pi$, so $e^{\pm j\beta l} = -1$. From Eq. (2.49a),

$$\Gamma = \frac{Z_{\text{L}} - Z_0}{Z_{\text{L}} + Z_0} = \frac{100 - 50}{100 + 50} = \frac{1}{3}.$$

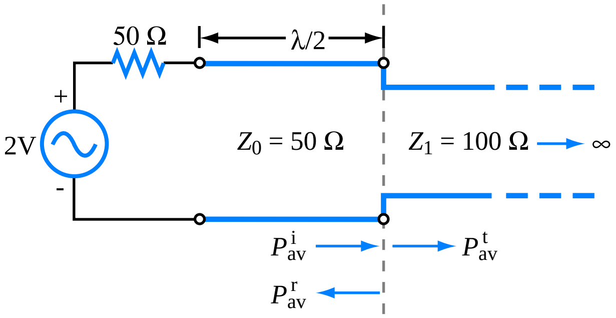


Figure P2.33: Line terminated in an infinite line.

Also, converting the generator to a phasor gives $\widetilde{V}_{\rm g} = 2e^{j0^\circ}$ (V). Plugging all these results into Eq. (2.66),

$$V_0^+ = \left(\frac{\widetilde{V}_{\rm g} Z_{\rm in}}{Z_{\rm g} + Z_{\rm in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) = \left(\frac{2\times 100}{50+100}\right)\left(\frac{1}{(-1)+\frac{1}{3}(-1)}\right)$$
$$= 1e^{j180^\circ} = -1 \quad \text{(V)}.$$

From Eqs. (2.84), (2.85), and (2.86),

$$P^{\rm i}_{\rm av} = \frac{|V_0^+|^2}{2Z_0} = \frac{|1e^{j180^\circ}|^2}{2\times 50} = 10.0 \text{ mW},$$
$$P^{\rm r}_{\rm av} = -|\Gamma|^2 P^{\rm i}_{\rm av} = -\left|\frac{1}{3}\right|^2 \times 10 \text{ mW} = -1.1 \text{ mW},$$
$$P^{\rm t}_{\rm av} = P_{\rm av} = P^{\rm i}_{\rm av} + P^{\rm r}_{\rm av} = 10.0 \text{ mW} - 1.1 \text{ mW} = 8.9 \text{ mW}.$$

Problem 2.34 An antenna with a load impedance $Z_{\rm L} = (75 + j25)$ Ω is connected to a transmitter through a 50-Ω lossless transmission line. If under matched conditions (50-Ω load), the transmitter can deliver 20 W to the load, how much power does it deliver to the antenna? Assume $Z_{\rm g} = Z_0$.

Solution: From Eqs. (2.66) and (2.61),

$$\begin{aligned}
V_0^+ &= \left(\frac{\widetilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) \\
&= \left(\frac{\widetilde{V}_g Z_0\left[(1+\Gamma e^{-j2\beta l})/(1-\Gamma e^{-j2\beta l})\right]}{Z_0 + Z_0\left[(1+\Gamma e^{-j2\beta l})/(1-\Gamma e^{-j2\beta l})\right]}\right)\left(\frac{e^{-j\beta l}}{1+\Gamma e^{-j2\beta l}}\right) \\
&= \frac{\widetilde{V}_g e^{-j\beta l}}{(1-\Gamma e^{-j2\beta l}) + (1+\Gamma e^{-j2\beta l})} \\
&= \frac{\widetilde{V}_g e^{-j\beta l}}{(1-\Gamma e^{-j2\beta l}) + (1+\Gamma e^{-j2\beta l})} = \tfrac{1}{2}\widetilde{V}_g e^{-j\beta l}.
\end{aligned}$$

Thus, in Eq. (2.86),

$$P_{av} = \frac{|V_0^+|^2}{2Z_0}(1-|\Gamma|^2) = \frac{|\frac{1}{2}\widetilde{V}_g e^{-j\beta l}|^2}{2Z_0}(1-|\Gamma|^2) = \frac{|\widetilde{V}_g|^2}{8Z_0}(1-|\Gamma|^2).$$

Under the matched condition, $|\Gamma| = 0$ and $P_L = 20$ W, so $|\widetilde{V}_g|^2/8Z_0 = 20$ W.

When $Z_L = (75 + j25)\ \Omega$, from Eq. (2.49a),

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(75+j25)\ \Omega - 50\ \Omega}{(75+j25)\ \Omega + 50\ \Omega} = 0.277e^{j33.6^\circ},$$

so $P_{av} = 20\text{ W}\,(1-|\Gamma|^2) = 20\text{ W}\,(1-0.277^2) = 18.46$ W.

Section 2-9: Smith Chart

Problem 2.35 Use the Smith chart to find the reflection coefficient corresponding to a load impedance:

(a) $Z_L = 3Z_0$,

(b) $Z_L = (2-2j)Z_0$,

(c) $Z_L = -2jZ_0$,

(d) $Z_L = 0$ (short circuit).

Solution: Refer to Fig. P2.35.

(a) Point A is $z_L = 3 + j0$. $\Gamma = 0.5e^{0^\circ}$

(b) Point B is $z_L = 2 - j2$. $\Gamma = 0.62e^{-29.7^\circ}$

(c) Point C is $z_L = 0 - j2$. $\Gamma = 1.0e^{-53.1^\circ}$

(d) Point D is $z_L = 0 + j0$. $\Gamma = 1.0e^{180.0^\circ}$

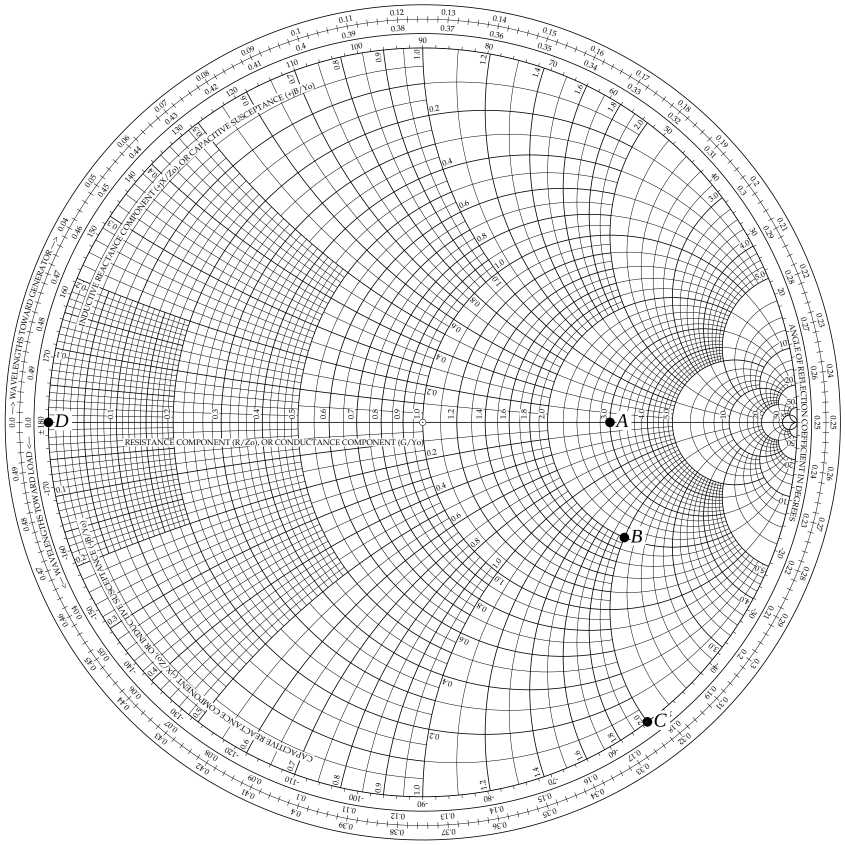

A
B
C
D
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD

Figure P2.35: Solution of Problem 2.35.

Problem 2.36 Use the Smith chart to find the normalized load impedance corresponding to a reflection coefficient:

(a) $\Gamma = 0.5$,

(b) $\Gamma = 0.5\underline{/60^\circ}$,

(c) $\Gamma = -1$,

(d) $\Gamma = 0.3\underline{/-30^\circ}$,

(e) $\Gamma = 0$,

(f) $\Gamma = j$.

Solution: Refer to Fig. P2.36.

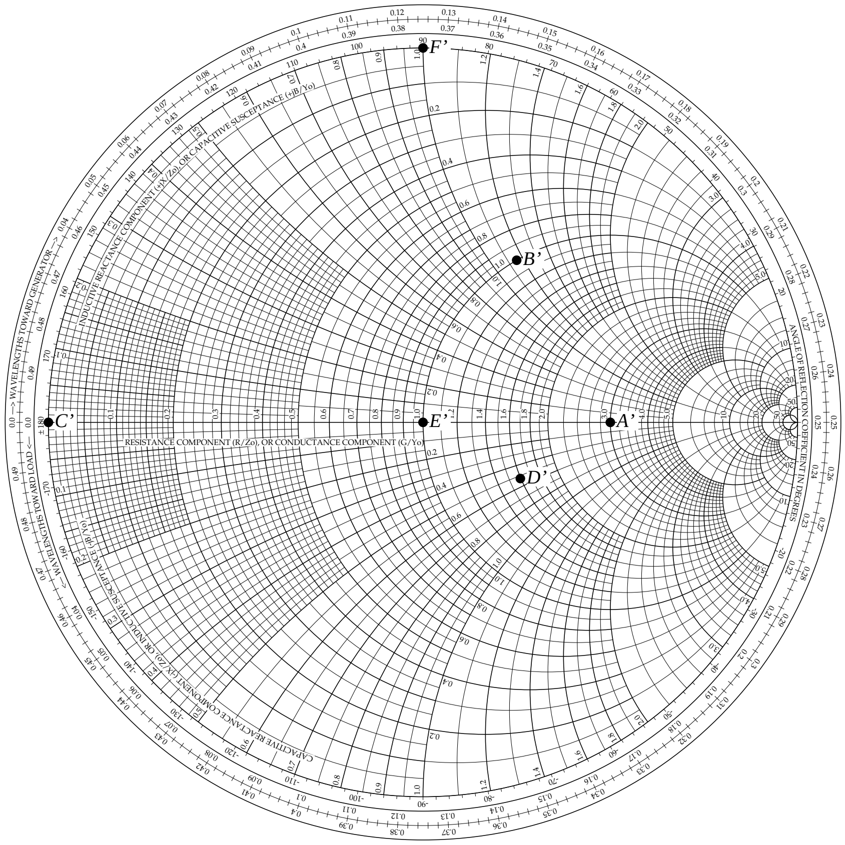

A'
B'
C'
D'
E'
F'
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Figure P2.36: Solution of Problem 2.36.

(a) Point A' is $\Gamma = 0.5$ at $z_L = 3 + j0$.
(b) Point B' is $\Gamma = 0.5e^{j60^\circ}$ at $z_L = 1 + j1.15$.
(c) Point C' is $\Gamma = -1$ at $z_L = 0 + j0$.
(d) Point D' is $\Gamma = 0.3e^{-j30^\circ}$ at $z_L = 1.60 - j0.53$.
(e) Point E' is $\Gamma = 0$ at $z_L = 1 + j0$.
(f) Point F' is $\Gamma = j$ at $z_L = 0 + j1$.

Problem 2.37 On a lossless transmission line terminated in a load $Z_L = 100\ \Omega$, the standing-wave ratio was measured to be 2.5. Use the Smith chart to find the two possible values of Z_0.

Solution: Refer to Fig. P2.37. $S = 2.5$ is at point $L1$ and the constant SWR circle is shown. $z_{\rm L}$ is real at only two places on the SWR circle, at $L1$, where $z_{\rm L} = S = 2.5$, and $L2$, where $z_{\rm L} = 1/S = 0.4$. so $Z_{01} = Z_{\rm L}/z_{\rm L1} = 100\ \Omega/2.5 = 40\ \Omega$ and $Z_{02} = Z_{\rm L}/z_{\rm L2} = 100\ \Omega/0.4 = 250\ \Omega$.

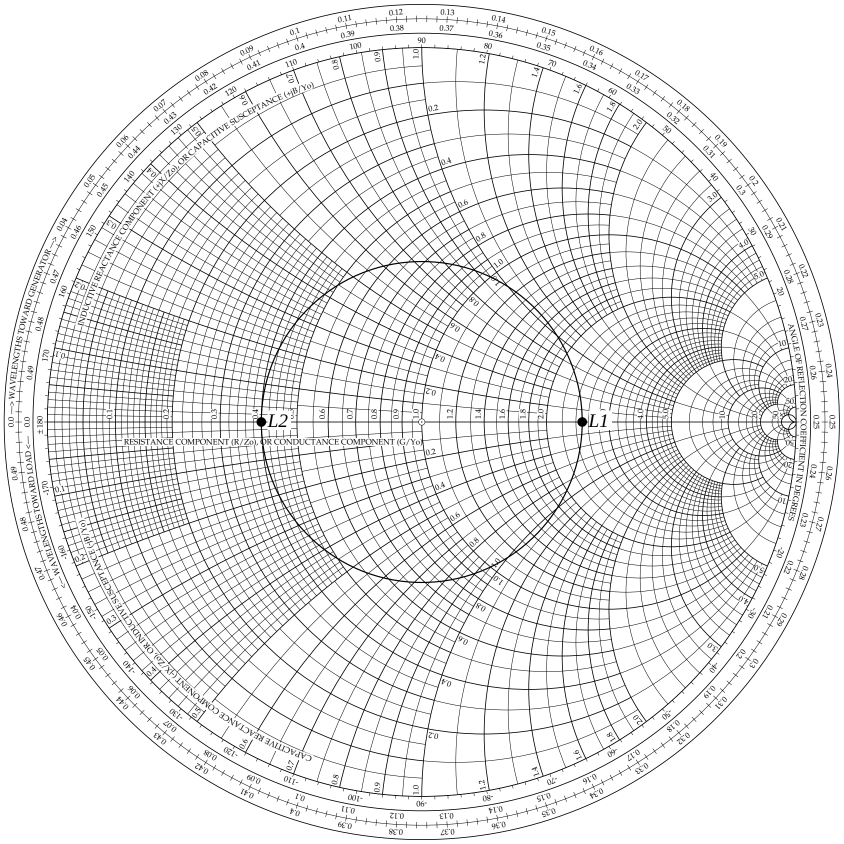

L2
L1
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Figure P2.37: Solution of Problem 2.37.

Problem 2.38 A lossless 50-Ω transmission line is terminated in a load with $Z_{\rm L} = (50 + j25)\ \Omega$. Use the Smith chart to find the following:

(a) the reflection coefficient Γ,

(b) the standing-wave ratio,

(c) the input impedance at 0.35λ from the load,

(d) the input admittance at 0.35λ from the load,

(e) the shortest line length for which the input impedance is purely resistive,

(f) the position of the first voltage maximum from the load.

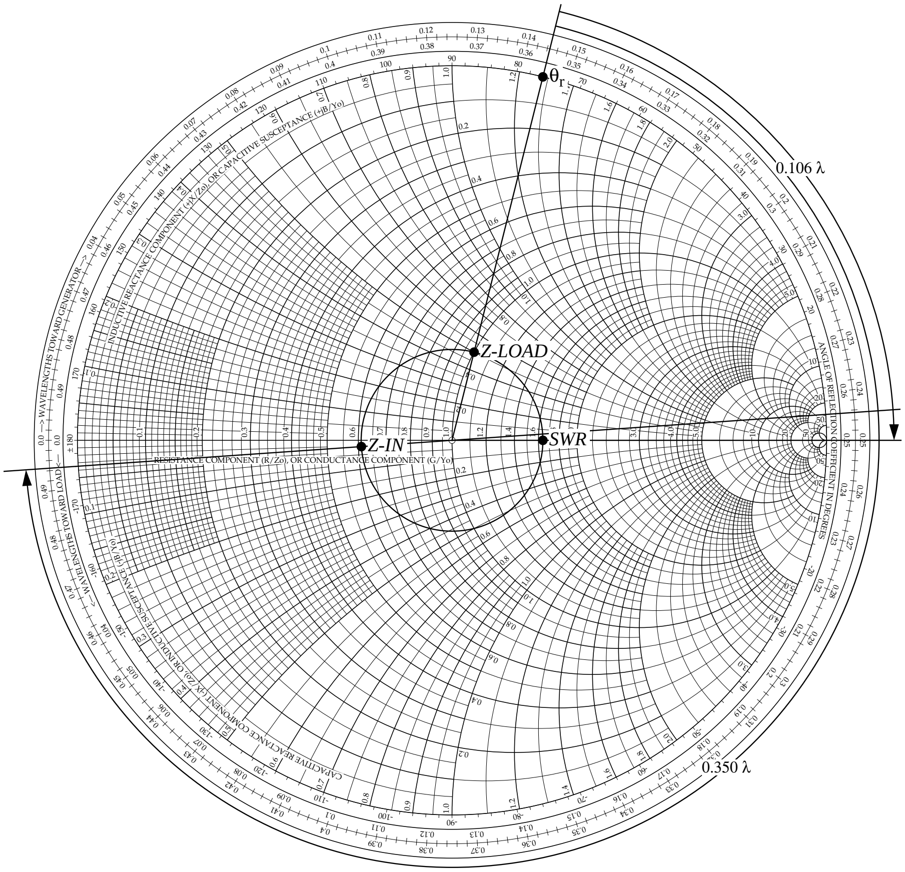

θr
0.106 λ
Z-LOAD
Z-IN
SWR
0.350 λ
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Figure P2.38: Solution of Problem 2.38.

Solution: Refer to Fig. P2.38. The normalized impedance

$$z_{\mathrm{L}} = \frac{(50 + j25)\ \Omega}{50\ \Omega} = 1 + j0.5$$

is at point *Z-LOAD*.

(a) $\Gamma = 0.24 e^{j76.0^\circ}$ The angle of the reflection coefficient is read of that scale at the point θ_{r}.

(b) At the point *SWR*: $S = 1.64$.

(c) Z_{in} is 0.350λ from the load, which is at 0.144λ on the wavelengths to generator scale. So point *Z-IN* is at $0.144\lambda + 0.350\lambda = 0.494\lambda$ on the WTG scale. At point *Z-IN*:

$$Z_{\text{in}} = z_{\text{in}} Z_0 = (0.61 - j0.022) \times 50\ \Omega = (30.5 - j1.09)\ \Omega.$$

(d) At the point on the SWR circle opposite *Z-IN*,

$$Y_{\text{in}} = \frac{y_{\text{in}}}{Z_0} = \frac{(1.64 + j0.06)}{50\ \Omega} = (32.7 + j1.17)\ \text{mS}.$$

(e) Traveling from the point *Z-LOAD* in the direction of the generator (clockwise), the SWR circle crosses the $x_{\text{L}} = 0$ line first at the point *SWR*. To travel from *Z-LOAD* to *SWR* one must travel $0.250\lambda - 0.144\lambda = 0.106\lambda$. (Readings are on the wavelengths to generator scale.) So the shortest line length would be 0.106λ.

(f) The voltage max occurs at point *SWR*. From the previous part, this occurs at $z = -0.106\lambda$.

Problem 2.39 A lossless 50-Ω transmission line is terminated in a short circuit. Use the Smith chart to find

(a) the input impedance at a distance 2.3λ from the load,

(b) the distance from the load at which the input admittance is $Y_{\text{in}} = -j0.04$ S.

Solution: Refer to Fig. P2.39.

(a) For a short, $z_{\text{in}} = 0 + j0$. This is point *Z-SHORT* and is at 0.000λ on the WTG scale. Since a lossless line repeats every $\lambda/2$, traveling 2.3λ toward the generator is equivalent to traveling 0.3λ toward the generator. This point is at A : *Z-IN*, and

$$Z_{\text{in}} = z_{\text{in}} Z_0 = (0 - j3.08) \times 50\ \Omega = -j154\ \Omega.$$

(b) The admittance of a short is at point *Y-SHORT* and is at 0.250λ on the WTG scale:

$$y_{\text{in}} = Y_{\text{in}} Z_0 = -j0.04\ \text{S} \times 50\ \Omega = -j2,$$

which is point B : *Y-IN* and is at 0.324λ on the WTG scale. Therefore, the line length is $0.324\lambda - 0.250\lambda = 0.074\lambda$. Any integer half wavelengths farther is also valid.

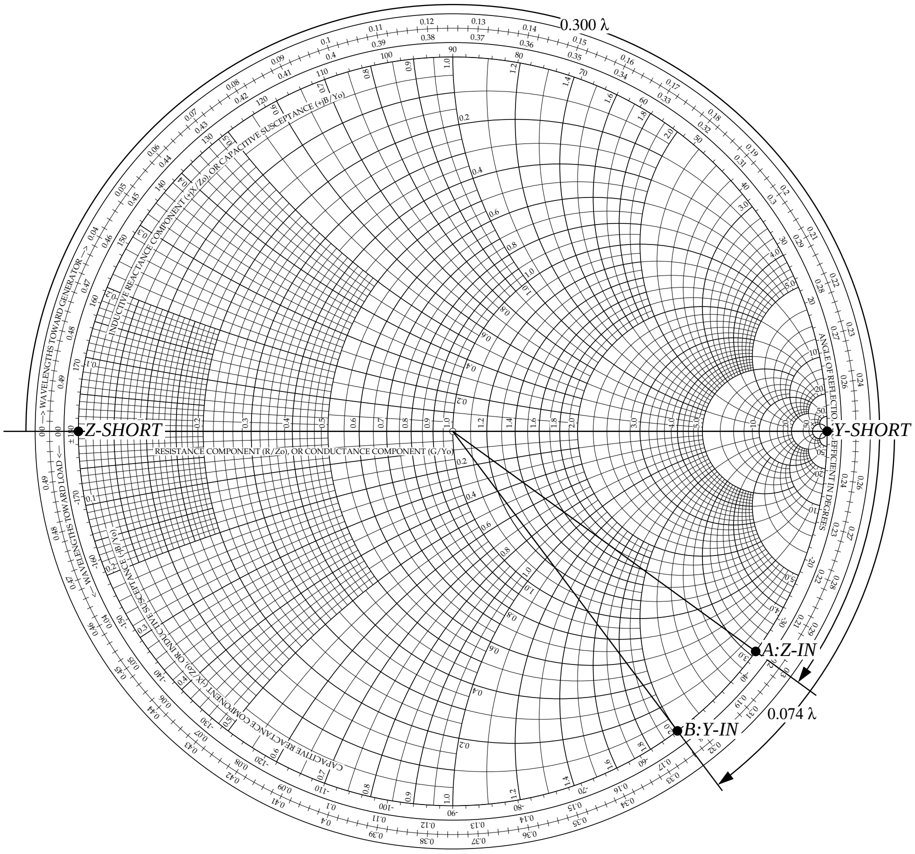

0.300 λ
Z-SHORT
Y-SHORT
A:Z-IN
B:Y-IN
0.074 λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Figure P2.39: Solution of Problem 2.39.

Problem 2.40 Use the Smith chart to find y_{L} if $z_{\mathrm{L}} = 1.5 - j0.7$.

Solution: Refer to Fig. P2.40. The point Z represents $1.5 - j0.7$. The reciprocal of point Z is at point Y, which is at $0.55 + j0.26$.

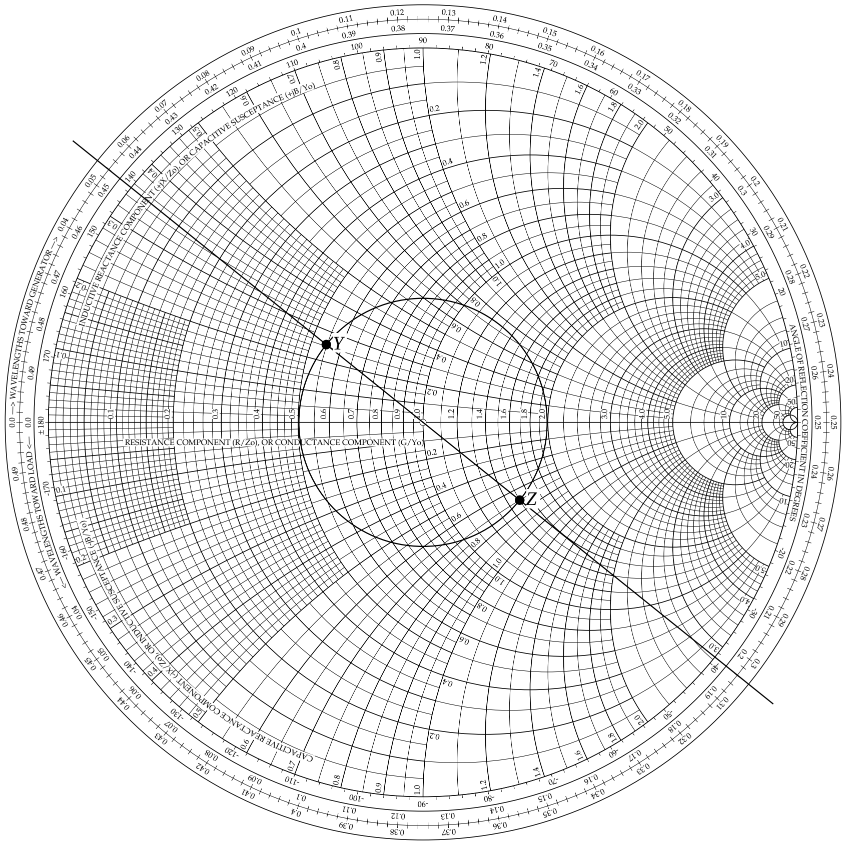

Y
Z
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD

Figure P2.40: Solution of Problem 2.40.

Problem 2.41 A lossless 100-Ω transmission line $3\lambda/8$ in length is terminated in an unknown impedance. If the input impedance is $Z_{\text{in}} = -j2.5$ Ω,

(a) use the Smith chart to find Z_{L}.

(b) What length of open-circuit line could be used to replace Z_{L}?

Solution: Refer to Fig. P2.41. $z_{\text{in}} = Z_{\text{in}}/Z_0 = -j2.5\ \Omega/100\ \Omega = 0.0 - j0.025$ which is at point *Z-IN* and is at 0.004λ on the wavelengths to load scale.

(a) Point *Z-LOAD* is 0.375λ toward the load from the end of the line. Thus, on the wavelength to load scale, it is at $0.004\lambda + 0.375\lambda = 0.379\lambda$.

$$Z_{\text{L}} = z_{\text{L}} Z_0 = (0 + j0.95) \times 100\ \Omega = j95\ \Omega.$$

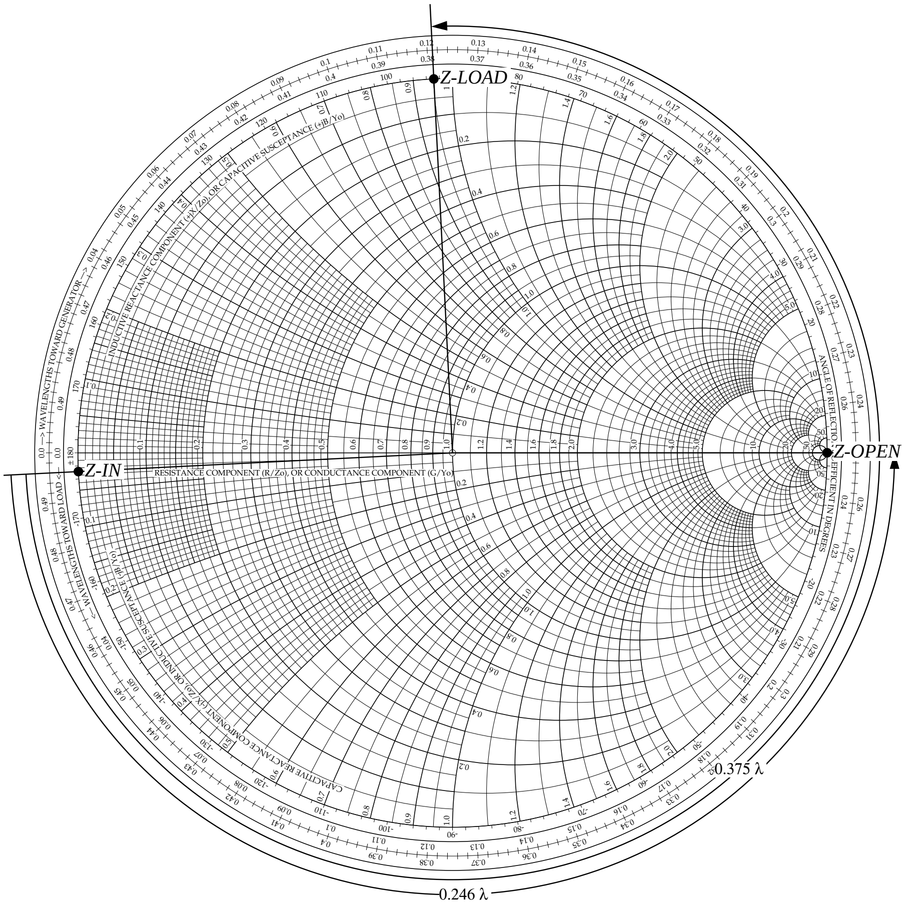


Figure P2.41: Solution of Problem 2.41.

(b) An open circuit is located at point *Z-OPEN*, which is at 0.250λ on the wavelength to load scale. Therefore, an open circuited line with $Z_{\text{in}} = -j0.025$ must have a length of $0.250\lambda - 0.004\lambda = 0.246\lambda$.

Problem 2.42 A 75-Ω lossless line is 0.6λ long. If $S = 1.8$ and $\theta_r = -60°$, use the Smith chart to find $|\Gamma|$, Z_L, and Z_{in}.

Solution: Refer to Fig. P2.42. The SWR circle must pass through $S = 1.8$ at point *SWR*. A circle of this radius has

$$|\Gamma| = \frac{S-1}{S+1} = 0.29.$$

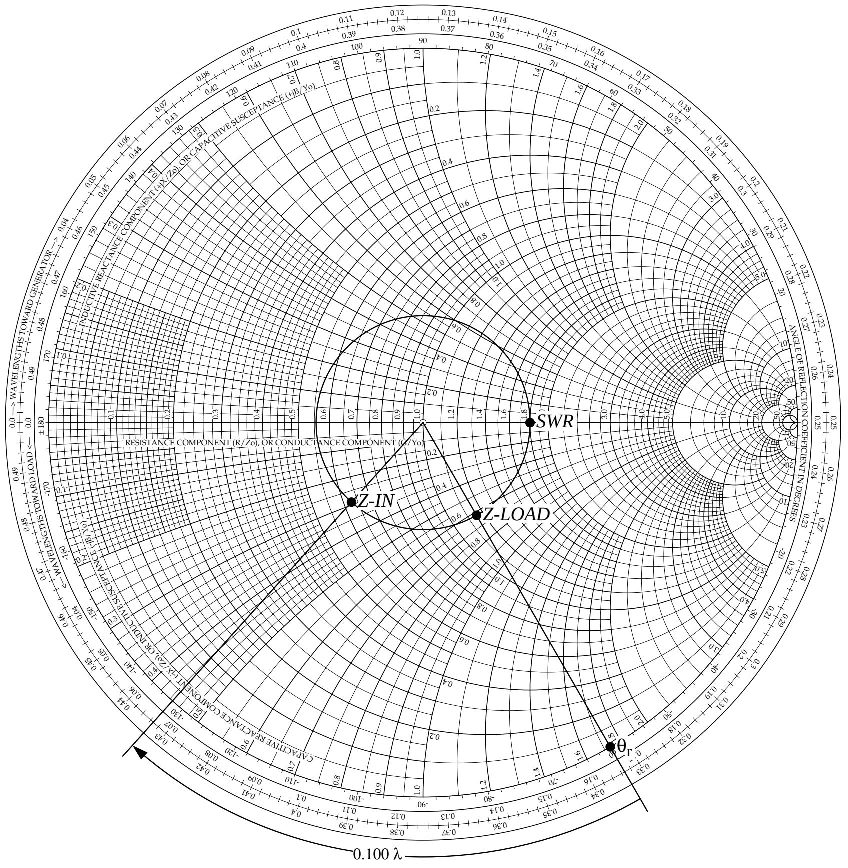

SWR
Z-IN
Z-LOAD
θ_r
0.100 λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD

Figure P2.42: Solution of Problem 2.42.

The load must have a reflection coefficient with $\theta_r = -60°$. The angle of the reflection coefficient is read off that scale at the point θ_r. The intersection of the circle of constant $|\Gamma|$ and the line of constant θ_r is at the load, point *Z-LOAD*, which has a value $z_L = 1.15 - j0.62$. Thus,

$$Z_L = z_L Z_0 = (1.15 - j0.62) \times 75\ \Omega = (86.5 - j46.6)\ \Omega.$$

A 0.6λ line is equivalent to a 0.1λ line. On the WTG scale, *Z-LOAD* is at 0.333λ, so *Z-IN* is at $0.333\lambda + 0.100\lambda = 0.433\lambda$ and has a value

$$z_{\text{in}} = 0.63 - j0.29.$$

Therefore $Z_{\text{in}} = z_{\text{in}} Z_0 = (0.63 - j0.29) \times 75\ \Omega = (47.0 - j21.8)\ \Omega$.

Problem 2.43 Using a slotted line on a 50-Ω air-spaced lossless line, the following measurements were obtained: $S = 1.6$, $|\widetilde{V}|_{\max}$ occurred only at 10 cm and 24 cm from the load. Use the Smith chart to find Z_{L}.

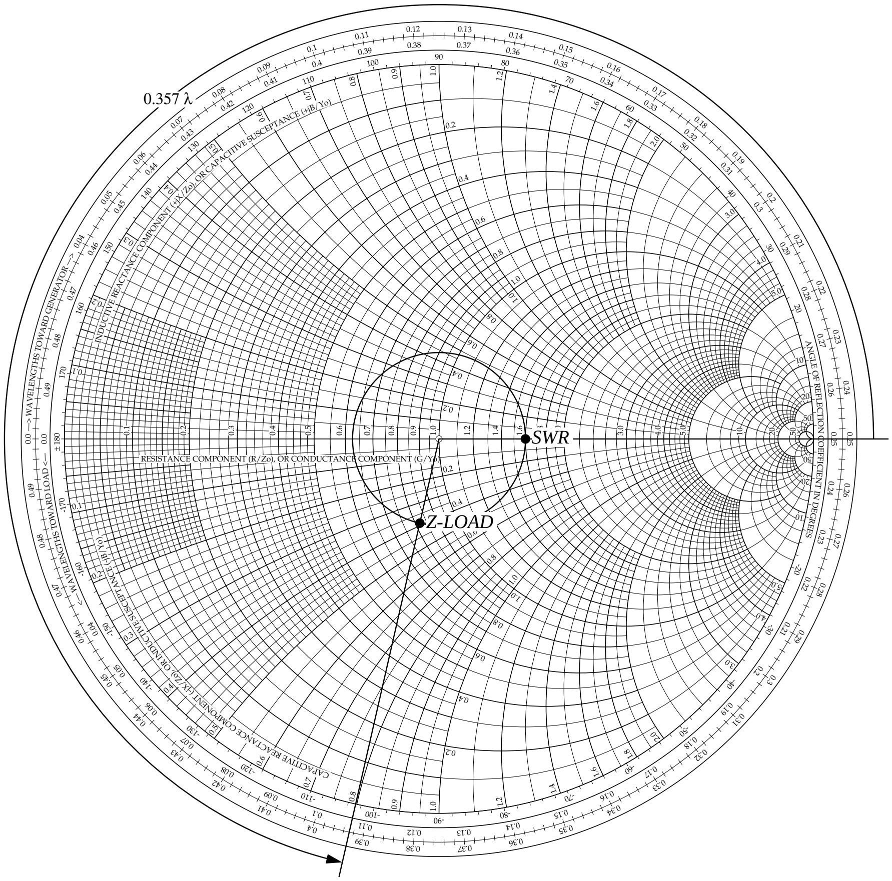

0.357 λ
SWR
Z-LOAD
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Figure P2.43: Solution of Problem 2.43.

Solution: Refer to Fig. P2.43. The point *SWR* denotes the fact that $S = 1.6$. This point is also the location of a voltage maximum. From the knowledge of the locations of adjacent maxima we can determine that $\lambda = 2(24\text{ cm} - 10\text{ cm}) = 28$ cm. Therefore, the load is $\frac{10\text{ cm}}{28\text{ cm}}\lambda = 0.357\lambda$ from the first voltage maximum, which is at 0.250λ on the WTL scale. Traveling this far on the SWR circle we find point *Z-LOAD* at $0.250\lambda + 0.357\lambda - 0.500\lambda = 0.107\lambda$ on the WTL scale, and here

$$z_L = 0.82 - j0.39.$$

Therefore $Z_L = z_L Z_0 = (0.82 - j0.39) \times 50\ \Omega = (41.0 - j19.5)\ \Omega$.

Problem 2.44 At an operating frequency of 5 GHz, a 50-Ω lossless coaxial line with insulating material having a relative permittivity $\varepsilon_r = 2.25$ is terminated in an antenna with an impedance $Z_L = 150\ \Omega$. Use the Smith chart to find Z_{in}. The line length is 30 cm.

Solution: To use the Smith chart the line length must be converted into wavelengths. Since $\beta = 2\pi/\lambda$ and $u_p = \omega/\beta$,

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi u_p}{\omega} = \frac{c}{\sqrt{\varepsilon_r} f} = \frac{3 \times 10^8 \text{ m/s}}{\sqrt{2.25} \times (5 \times 10^9 \text{ Hz})} = 0.04 \text{ m}.$$

Hence, $l = \frac{0.30 \text{ m}}{0.04 \text{ m}}\lambda = 7.5\lambda$. Since this is an integral number of half wavelengths,

$$Z_{in} = Z_L = 150\ \Omega.$$

Section 2-10: Impedance Matching

Problem 2.45 A 50-Ω lossless line 0.6λ long is terminated in a load with $Z_L = (50 + j25)\ \Omega$. At 0.3λ from the load, a resistor with resistance $R = 30\ \Omega$ is connected as shown in Fig. 2-43 (P2.45(a)). Use the Smith chart to find Z_{in}.

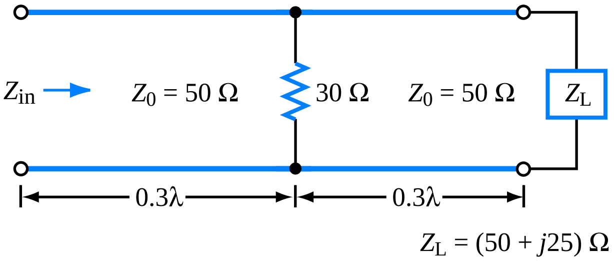


Figure P2.45: (a) Circuit for Problem 2.45.

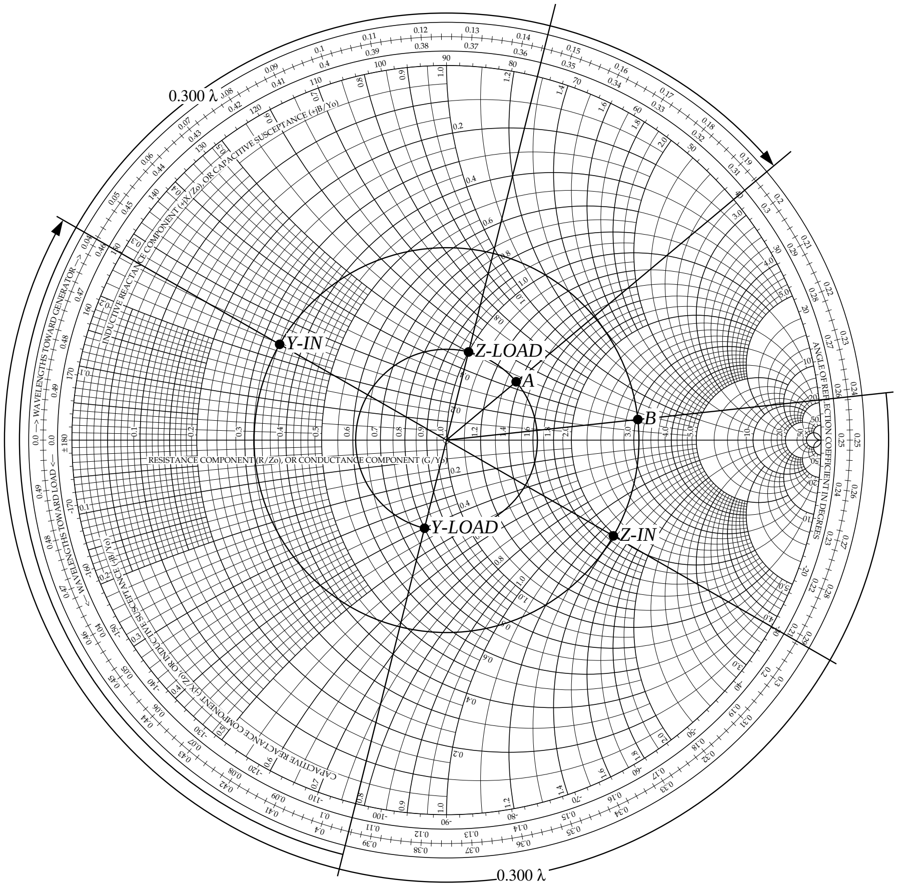

0.300 λ
Y-IN
Z-LOAD
A
B
Y-LOAD
Z-IN
0.300 λ
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Figure P2.45: (b) Solution of Problem 2.45.

Solution: Refer to Fig. P2.45(b). Since the 30-Ω resistor is in parallel with the input impedance at that point, it is advantageous to convert all quantities to admittances.

$$z_{\mathrm{L}} = \frac{Z_{\mathrm{L}}}{Z_0} = \frac{(50 + j25)\ \Omega}{50\ \Omega} = 1 + j0.5$$

and is located at point *Z-LOAD*. The corresponding normalized load admittance is at point *Y-LOAD*, which is at 0.394λ on the WTG scale. The input admittance of the load only at the shunt conductor is at $0.394\lambda + 0.300\lambda - 0.500\lambda = 0.194\lambda$ and is denoted by point *A*. It has a value of

$$y_{\mathrm{in}A} = 1.37 + j0.45.$$

The shunt conductance has a normalized conductance

$$g = \frac{50\ \Omega}{30\ \Omega} = 1.67.$$

The normalized admittance of the shunt conductance in parallel with the input admittance of the load is the sum of their admittances:

$$y_{\text{in}B} = g + y_{\text{in}A} = 1.67 + 1.37 + j0.45 = 3.04 + j0.45$$

and is located at point B. On the WTG scale, point B is at 0.242λ. The input admittance of the entire circuit is at $0.242\lambda + 0.300\lambda - 0.500\lambda = 0.042\lambda$ and is denoted by point *Y-IN*. The corresponding normalized input impedance is at *Z-IN* and has a value of

$$z_{\text{in}} = 1.9 - j1.4.$$

Thus,

$$Z_{\text{in}} = z_{\text{in}} Z_0 = (1.9 - j1.4) \times 50\ \Omega = (95 - j70)\ \Omega.$$

Problem 2.46 A 50-Ω lossless line is to be matched to an antenna with

$$Z_{\text{L}} = (75 - j20)\ \Omega$$

using a shorted stub. Use the Smith chart to determine the stub length and the distance between the antenna and the stub.

Solution: Refer to Fig. P2.46(a) and Fig. P2.46(b), which represent two different solutions.

$$z_{\text{L}} = \frac{Z_{\text{L}}}{Z_0} = \frac{(75 - j20)\ \Omega}{50\ \Omega} = 1.5 - j0.4$$

and is located at point *Z-LOAD* in both figures. Since it is advantageous to work in admittance coordinates, y_{L} is plotted as point *Y-LOAD* in both figures. *Y-LOAD* is at 0.041λ on the WTG scale.

For the first solution in Fig. P2.46(a), point *Y-LOAD-IN*-1 represents the point at which $g = 1$ on the SWR circle of the load. *Y-LOAD-IN*-1 is at 0.145λ on the WTG scale, so the stub should be located at $0.145\lambda - 0.041\lambda = 0.104\lambda$ from the load (or some multiple of a half wavelength further). At *Y-LOAD-IN*-1, $b = 0.52$, so a stub with an input admittance of $y_{\text{stub}} = 0 - j0.52$ is required. This point is *Y-STUB-IN*-1 and is at 0.423λ on the WTG scale. The short circuit admittance

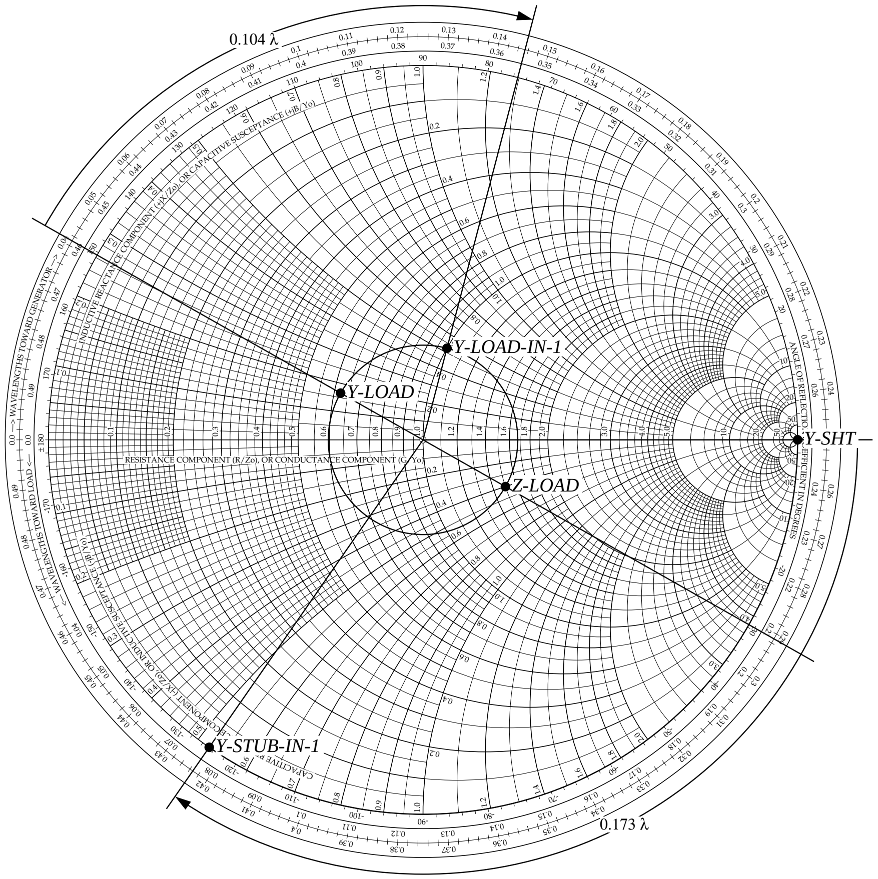

0.104 λ
Y-LOAD-IN-1
Y-LOAD
Y-SHT
Z-LOAD
Y-STUB-IN-1
0.173 λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Figure P2.46: (a) First solution to Problem 2.46.

is denoted by point *Y*-*SHT*, located at 0.250λ. Therefore, the short stub must be 0.423λ − 0.250λ = 0.173λ long (or some multiple of a half wavelength longer).

For the second solution in Fig. P2.46(b), point *Y*-*LOAD*-*IN*-2 represents the point at which $g = 1$ on the SWR circle of the load. *Y*-*LOAD*-*IN*-2 is at 0.355λ on the WTG scale, so the stub should be located at 0.355λ − 0.041λ = 0.314λ from the load (or some multiple of a half wavelength further). At *Y*-*LOAD*-*IN*-2, $b = -0.52$, so a stub with an input admittance of $y_{\text{stub}} = 0 + j0.52$ is required. This point is *Y*-*STUB*-*IN*-2 and is at 0.077λ on the WTG scale. The short circuit admittance is denoted by point *Y*-*SHT*, located at 0.250λ. Therefore, the short stub must be 0.077λ − 0.250λ + 0.500λ = 0.327λ long (or some multiple of a half wavelength

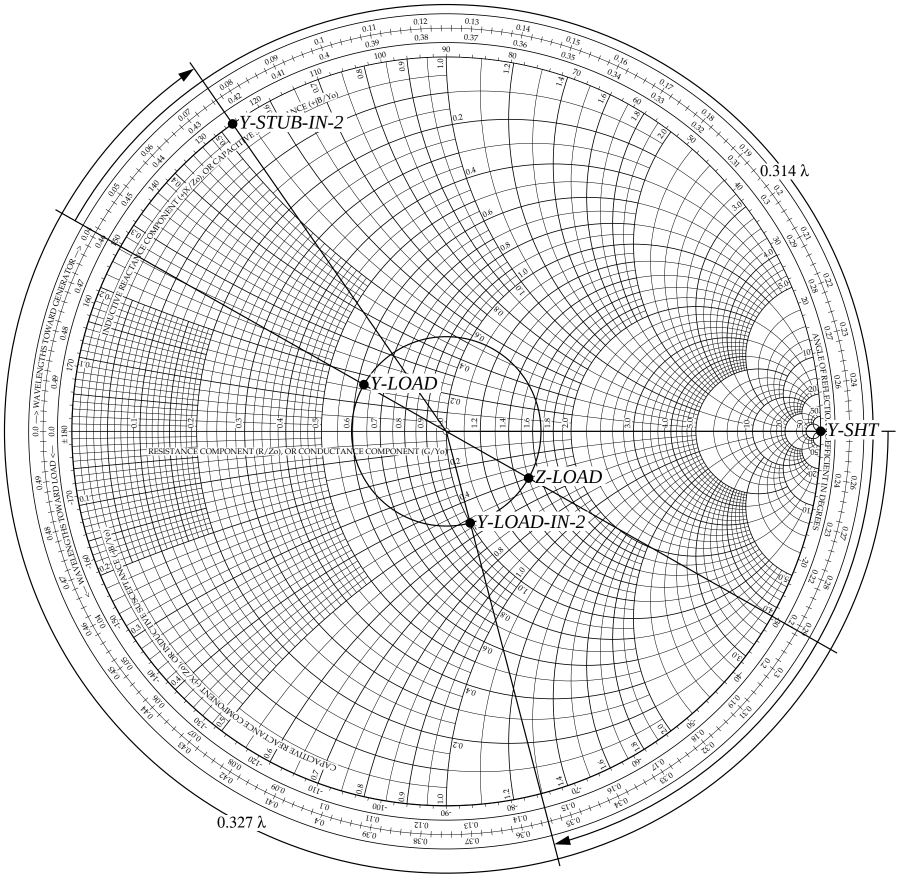

Y-STUB-IN-2
0.314 λ
Y-LOAD
Y-SHT
Z-LOAD
Y-LOAD-IN-2
0.327 λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Figure P2.46: (b) Second solution to Problem 2.46.

longer).

Problem 2.47 Repeat Problem 2.46 for a load with $Z_L = (100 + j50)\ \Omega$.

Solution: Refer to Fig. P2.47(a) and Fig. P2.47(b), which represent two different solutions.

$$z_L = \frac{Z_L}{Z_0} = \frac{100 + j50\ \Omega}{50\ \Omega} = 2 + j1$$

and is located at point *Z-LOAD* in both figures. Since it is advantageous to work in admittance coordinates, y_L is plotted as point *Y-LOAD* in both figures. *Y-LOAD* is at 0.463λ on the WTG scale.

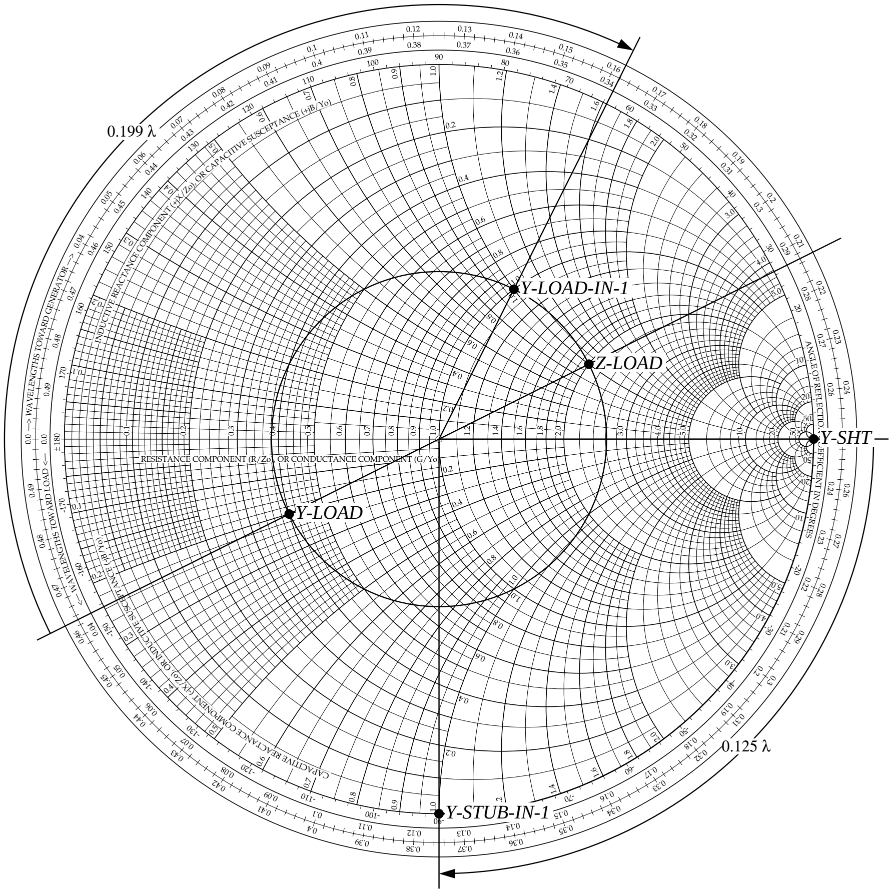

0.199 λ
Y-LOAD-IN-1
Z-LOAD
Y-SHT
Y-LOAD
Y-STUB-IN-1
0.125 λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo) OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo) OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
0.0 —> WAVELENGTHS TOWARD GENERATOR —>
<— WAVELENGTHS TOWARD LOAD <—

Figure P2.47: (a) First solution to Problem 2.47.

For the first solution in Fig. P2.47(a), point *Y-LOAD-IN*-1 represents the point at which $g = 1$ on the SWR circle of the load. *Y-LOAD-IN*-1 is at 0.162λ on the WTG scale, so the stub should be located at $0.162\lambda - 0.463\lambda + 0.500\lambda = 0.199\lambda$ from the load (or some multiple of a half wavelength further). At *Y-LOAD-IN*-1, $b = 1$, so a stub with an input admittance of $y_{\text{stub}} = 0 - j1$ is required. This point is *Y-STUB-IN*-1 and is at 0.375λ on the WTG scale. The short circuit admittance is denoted by point *Y-SHT*, located at 0.250λ. Therefore, the short stub must be $0.375\lambda - 0.250\lambda = 0.125\lambda$ long (or some multiple of a half wavelength longer).

For the second solution in Fig. P2.47(b), point *Y-LOAD-IN*-2 represents the point at which $g = 1$ on the SWR circle of the load. *Y-LOAD-IN*-2 is at 0.338λ on the

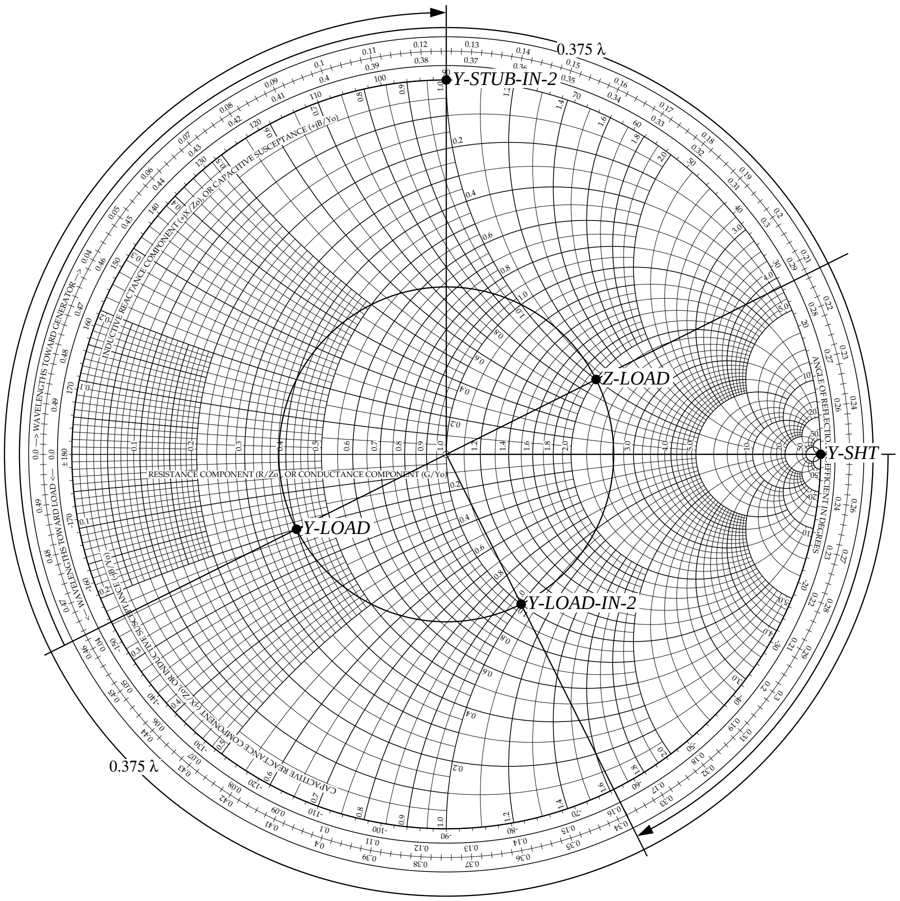

0.375 λ
Y-STUB-IN-2
Z-LOAD
Y-SHT
Y-LOAD
Y-LOAD-IN-2
0.375 λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Figure P2.47: (b) Second solution to Problem 2.47.

WTG scale, so the stub should be located at $0.338\lambda - 0.463\lambda + 0.500\lambda = 0.375\lambda$ from the load (or some multiple of a half wavelength further). At *Y-LOAD-IN*-2, $b = -1$, so a stub with an input admittance of $y_{\text{stub}} = 0 + j1$ is required. This point is *Y-STUB-IN*-2 and is at 0.125λ on the WTG scale. The short circuit admittance is denoted by point *Y-SHT*, located at 0.250λ. Therefore, the short stub must be $0.125\lambda - 0.250\lambda + 0.500\lambda = 0.375\lambda$ long (or some multiple of a half wavelength longer).

Problem 2.48 Use the Smith chart to find Z_{in} of the feed line shown in Fig. 2-44 (P2.48(a)). All lines are lossless with $Z_0 = 50\ \Omega$.

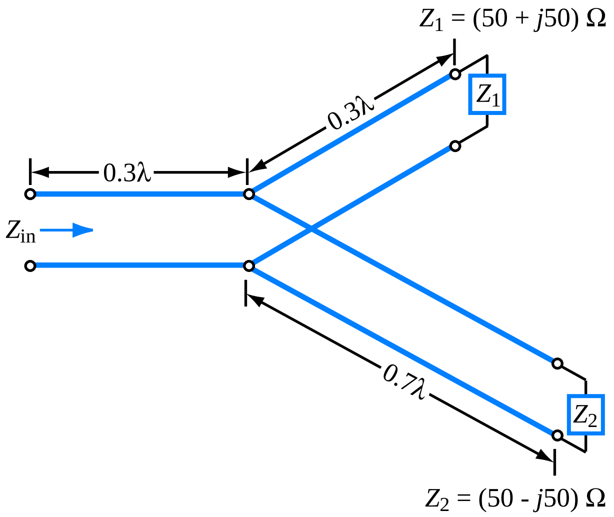


Figure P2.48: (a) Circuit of Problem 2.48.

Solution: Refer to Fig. P2.48(b).

$$z_1 = \frac{Z_1}{Z_0} = \frac{50 + j50\ \Omega}{50\ \Omega} = 1 + j1$$

and is at point *Z-LOAD*-1.

$$z_2 = \frac{Z_2}{Z_0} = \frac{50 - j50\ \Omega}{50\ \Omega} = 1 - j1$$

and is at point *Z-LOAD*-2. Since at the junction the lines are in parallel, it is advantageous to solve the problem using admittances. y_1 is point *Y-LOAD*-1, which is at 0.412λ on the WTG scale. y_2 is point *Y-LOAD*-2, which is at 0.088λ on the WTG scale. Traveling 0.300λ from *Y-LOAD*-1 toward the generator one obtains the input admittance for the upper feed line, point *Y-IN*-1, with a value of $1.97 + j1.02$. Since traveling 0.700λ is equivalent to traveling 0.200λ on any transmission line, the input admittance for the lower line feed is found at point *Y-IN*-2, which has a value of $1.97 - j1.02$. The admittance of the two lines together is the sum of their admittances: $1.97 + j1.02 + 1.97 - j1.02 = 3.94 + j0$ and is denoted *Y-JUNCT*. 0.300λ from *Y-JUNCT* toward the generator is the input admittance of the entire feed line, point *Y-IN*, from which *Z-IN* is found.

$$Z_{\text{in}} = z_{\text{in}} Z_0 = (1.65 - j1.79) \times 50\ \Omega = (82.5 - j89.5)\ \Omega.$$

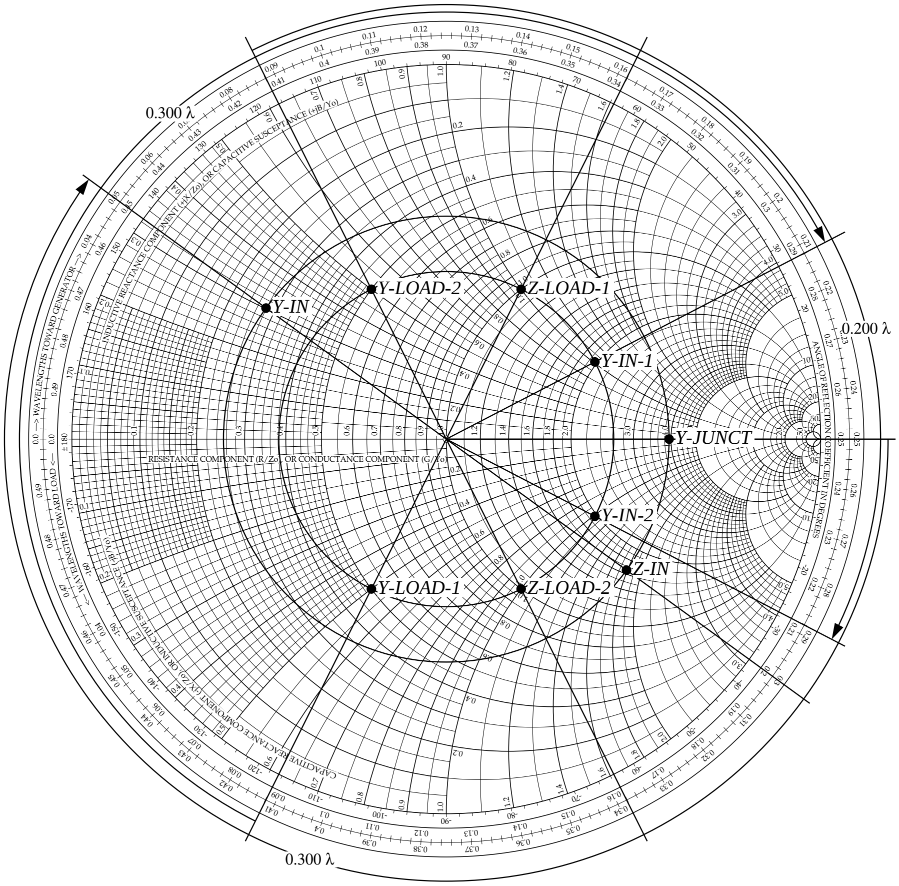

0.300 λ
0.200 λ
Y-IN
Y-LOAD-2
Z-LOAD-1
Y-IN-1
Y-JUNCT
Y-IN-2
Z-IN
Y-LOAD-1
Z-LOAD-2
0.300 λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Figure P2.48: (b) Solution of Problem 2.48.

Problem 2.49 Repeat Problem 2.48 for the case where all three transmission lines are $\lambda/4$ in length.

Solution: Since the transmission lines are in parallel, it is advantageous to express loads in terms of admittances. In the upper branch, which is a quarter wave line,

$$Y_{1\ \text{in}} = \frac{Y_0^2}{Y_1} = \frac{Z_1}{Z_0^2}\,,$$

and similarly for the lower branch,

$$Y_{2\,\mathrm{in}} = \frac{Y_0^2}{Y_2} = \frac{Z_2}{Z_0^2}\,.$$

Thus, the total load at the junction is

$$Y_{\mathrm{JCT}} = Y_{1\,\mathrm{in}} + Y_{2\,\mathrm{in}} = \frac{Z_1 + Z_2}{Z_0^2}\,.$$

Therefore, since the common transmission line is also quarter-wave,

$$Z_{\mathrm{in}} = Z_0^2/Z_{\mathrm{JCT}} = Z_0^2 Y_{\mathrm{JCT}} = Z_1 + Z_2 = (50 + j50)\ \Omega + (50 - j50)\ \Omega = 100\ \Omega.$$

Section 2-11: Transients on Transmission Lines

Problem 2.50 Generate a bounce diagram for the voltage $V(z,t)$ for a 1-m long lossless line characterized by $Z_0 = 50\ \Omega$ and $u_{\mathrm{p}} = 2c/3$ (where c is the velocity of light) if the line is fed by a step voltage applied at $t = 0$ by a generator circuit with $V_{\mathrm{g}} = 60$ V and $R_{\mathrm{g}} = 100\ \Omega$. The line is terminated in a load $Z_{\mathrm{L}} = 25\ \Omega$. Use the bounce diagram to plot $V(t)$ at a point midway along the length of the line from $t = 0$ to $t = 25$ ns.

Solution:

$$\Gamma_{\mathrm{g}} = \frac{R_{\mathrm{g}} - Z_0}{R_{\mathrm{g}} + Z_0} = \frac{100 - 50}{100 + 50} = \frac{50}{150} = \frac{1}{3}\,,$$

$$\Gamma_{\mathrm{L}} = \frac{Z_{\mathrm{L}} - Z_0}{Z_{\mathrm{L}} + Z_0} = \frac{25 - 50}{25 + 50} = \frac{-25}{75} = \frac{-1}{3}\,.$$

From Eq. (2.124b),

$$V_1^+ = \frac{V_{\mathrm{g}} Z_0}{R_{\mathrm{g}} + Z_0} = \frac{60 \times 50}{100 + 50} = 20\ \mathrm{V}.$$

Also,

$$T = \frac{l}{u_{\mathrm{p}}} = \frac{l}{2c/3} = \frac{3}{2 \times 3 \times 10^8} = 5\ \mathrm{ns}.$$

The bounce diagram is shown in Fig. P2.50(a) and the plot of $V(t)$ in Fig. P2.50(b).

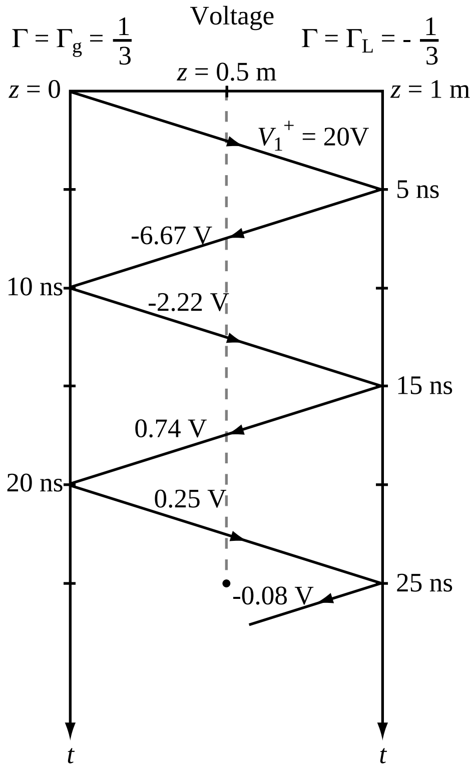


Figure P2.50: (a) Bounce diagram for Problem 2.50.

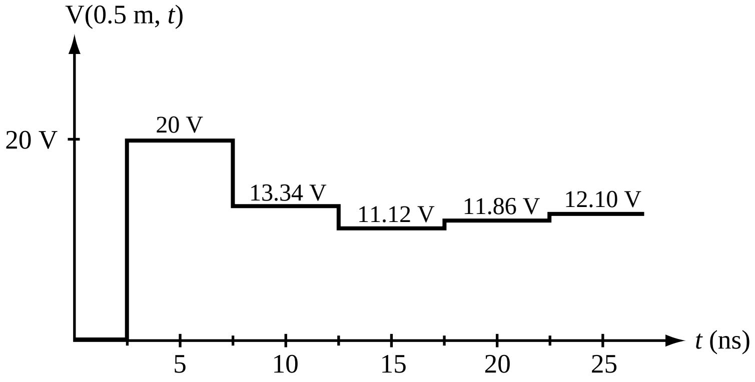


Figure P2.50: (b) Time response of voltage.

Problem 2.51 Repeat Problem 2.50 for the current I on the line.

Solution:

$$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} = \frac{100-50}{100+50} = \frac{1}{3},$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{25-50}{25+50} = \frac{-1}{3}.$$

From Eq. (2.124a),

$$I_1^+ = \frac{V_g}{R_g + Z_0} = \frac{60}{100+50} = 0.4 \text{ A}.$$

The bounce diagram is shown in Fig. P2.51(a) and $I(t)$ in Fig. P2.51(b).

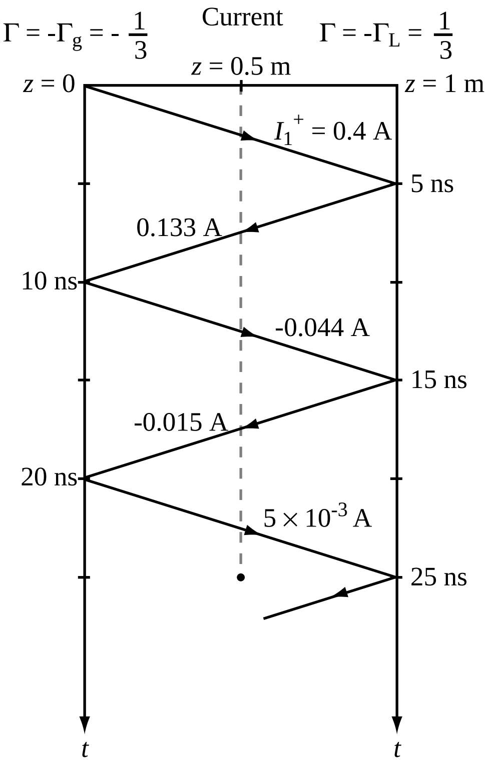


Figure P2.51: (a) Bounce diagram for Problem 2.51.

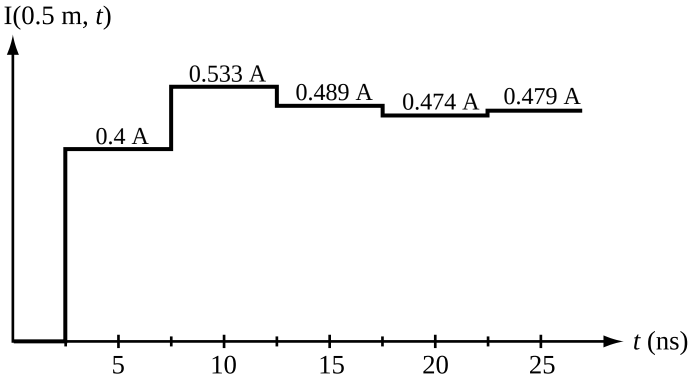


Figure P2.51: (b) Time response of current.

Problem 2.52 In response to a step voltage, the voltage waveform shown in Fig. 2-45 (P2.52) was observed at the sending end of a lossless transmission line with $R_g = 50\ \Omega$, $Z_0 = 50\ \Omega$, and $\varepsilon_r = 2.25$. Determine (a) the generator voltage, (b) the length of the line, and (c) the load impedance.

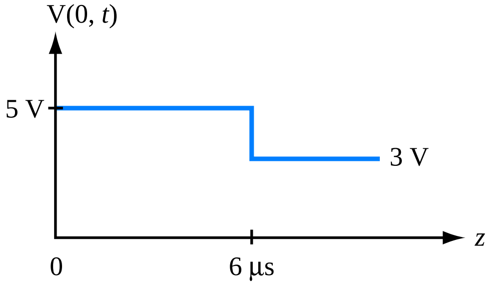


Figure P2.52: Observed voltage at sending end.

Solution:

(a) From the figure, $V_1^+ = 5$ V. Applying Eq. (2.124b),

$$V_1^+ = \frac{V_g Z_0}{R_g + Z_0} = \frac{V_g Z_0}{Z_0 + Z_0} = \frac{V_g}{2},$$

which gives $V_g = 2V_1^+ = 10$ V.

(b) $u_\text{p} = \dfrac{c}{\sqrt{\varepsilon_r}} = \dfrac{3\times 10^8}{\sqrt{2.25}} = 2\times 10^8$ m/s. The first change in the waveform occurs at $\Delta t = 6$ μs. But $\Delta t = 2l/u_\text{p}$. Hence,

$$l = \frac{\Delta t \mu_\text{p}}{2} = \frac{6\times 10^{-6}}{2} \times 2\times 10^8 = 600 \text{ m}.$$

(c) Since $R_\text{g} = Z_0$, $\Gamma_\text{g} = 0$. Hence $V_2^+ = 0$ and the change in level from 5 V down to 3 V is due to $V_1^- = -2$ V. But

$$V_1^- = \Gamma_\text{L} V_1^+, \qquad \text{or} \quad \Gamma_\text{L} = \frac{V_1^-}{V_1^+} = \frac{-2}{5} = -0.4.$$

From

$$Z_\text{L} = Z_0 \left(\frac{1+\Gamma_\text{L}}{1-\Gamma_\text{L}}\right) = 50 \left(\frac{1-0.4}{1+0.4}\right) = 21.43\ \Omega.$$

Problem 2.53 In response to a step voltage, the voltage waveform shown in Fig. 2.46 (P2.53) was observed at the sending end of a shorted line with $Z_0 = 50\ \Omega$ and $\varepsilon_\text{r} = 4$. Determine V_g, R_g, and the line length.

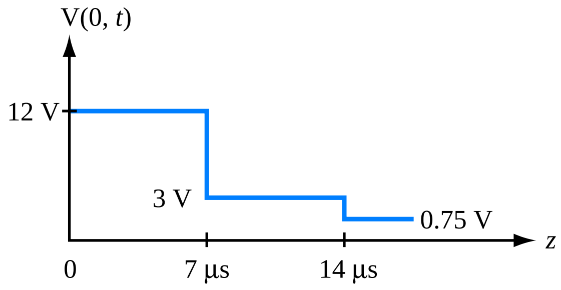


Figure P2.53: Observed voltage at sending end.

Solution:

$$u_\text{p} = \frac{c}{\sqrt{\varepsilon_\text{r}}} = \frac{3\times 10^8}{\sqrt{4}} = 1.5\times 10^8 \text{ m/s},$$

$$7\ \mu\text{s} = 7\times 10^{-6}\ \text{s} = \frac{2l}{u_\text{p}} = \frac{2l}{1.5\times 10^8}\,.$$

Hence, $l = 525$ m.

From the voltage waveform, $V_1^+ = 12$ V. At $t = 7\mu$s, the voltage at the sending end is

$$V(z=0, t=7\mu\text{s}) = V_1^+ + \Gamma_\text{L} V_1^+ + \Gamma_\text{g}\Gamma_\text{L} V_1^+ = -\Gamma_\text{g} V_1^+ \qquad \text{(because } \Gamma_\text{L} = -1).$$

Hence, 3 V$= -\Gamma_\text{g} \times 12$ V, or $\Gamma_\text{g} = -0.25$. From Eq. (2.128),

$$R_\text{g} = Z_0\left(\frac{1+\Gamma_\text{g}}{1-\Gamma_\text{g}}\right) = 50\left(\frac{1-0.25}{1+0.25}\right) = 30\ \Omega.$$

Also,

$$V_1^+ = \frac{V_\text{g} Z_0}{R_\text{g} + Z_0}, \qquad \text{or} \quad 12 = \frac{V_\text{g} \times 50}{30+50},$$

which gives $V_\text{g} = 19.2$ V.

Problem 2.54 Suppose the voltage waveform shown in Fig. 2-45 was observed at the sending end of a 50-Ω transmission line in response to a step voltage introduced by a generator with $V_\text{g} = 15$ V and an unknown series resistance R_g. The line is 1 km in length, its velocity of propagation is 1×10^8 m/s, and it is terminated in a load $Z_\text{L} = 100\ \Omega$.

(a) Determine R_g.

(b) Explain why the drop in level of $V(0,t)$ at $t = 6\ \mu$s cannot be due to reflection from the load.

(c) Determine the shunt resistance R_f and the location of the fault responsible for the observed waveform.

Solution:

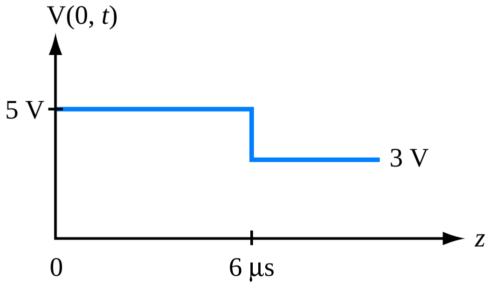


Figure P2.54: Observed voltage at sending end.

(a)

$$V_1^+ = \frac{V_g Z_0}{R_g + Z_0}.$$

From Fig. 2-45, $V_1^+ = 5$ V. Hence,

$$5 = \frac{15 \times 50}{R_g + 50},$$

which gives $R_g = 100\ \Omega$ and $\Gamma_g = 1/3$.

(b) Roundtrip time delay of pulse return from the load is

$$2T = \frac{2l}{u_p} = \frac{2 \times 10^3}{1 \times 10^8} = 20\ \mu\text{s},$$

which is much longer than 6 μs, the instance at which $V(0,t)$ drops in level.

(c) The new level of 3 V is equal to V_1^+ plus V_1^- plus V_2^+,

$$V_1^+ + V_1^- + V_2^+ = 5 + 5\Gamma_f + 5\Gamma_f\Gamma_g = 3 \quad \text{(V)},$$

which yields $\Gamma_f = -0.3$. But

$$\Gamma_f = \frac{Z_{Lf} - Z_0}{Z_{Lf} + Z_0} = -0.3,$$

which gives $Z_{Lf} = 26.92\ \Omega$. Since Z_{Lf} is equal to R_f and Z_0 in parallel, $R_f = 58.33\ \Omega$.

Problem 2.55 A generator circuit with $V_g = 200$ V and $R_g = 25\ \Omega$ was used to excite a 75-Ω lossless line with a rectangular pulse of duration $\tau = 0.4\ \mu$s. The line is 200 m long, its $u_p = 2 \times 10^8$ m/s, and it is terminated in a load $Z_L = 125\ \Omega$.

(a) Synthesize the voltage pulse exciting the line as the sum of two step functions, $V_{g_1}(t)$ and $V_{g_2}(t)$.

(b) For each voltage step function, generate a bounce diagram for the voltage on the line.

(c) Use the bounce diagrams to plot the total voltage at the sending end of the line.

Solution:

(a) pulse length $= 0.4\ \mu$s.

$$V_g(t) = V_{g_1}(t) + V_{g_2}(t),$$

with

$$V_{g_1}(t) = 200U(t) \quad \text{(V)},$$
$$V_{g_2}(t) = -200U(t - 0.4\ \mu\text{s}) \quad \text{(V)}.$$

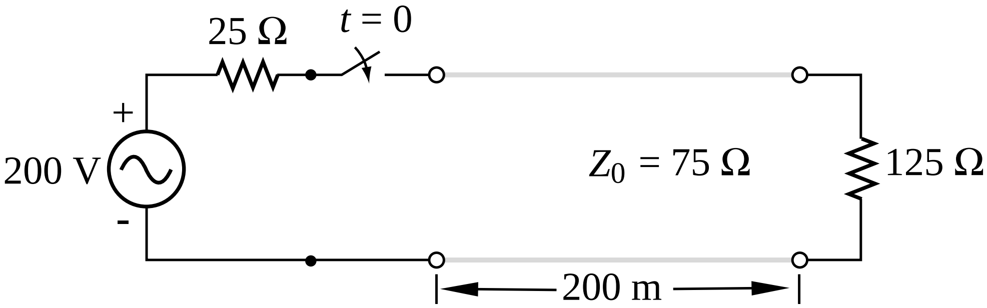


Figure P2.55: (a) Circuit for Problem 2.55.

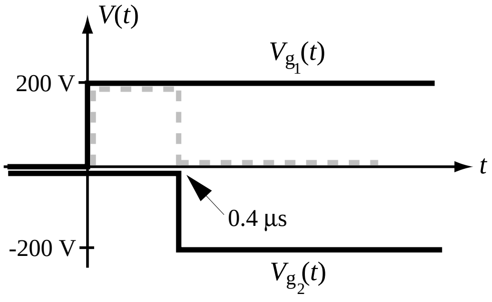


Figure P2.55: (b) Solution of part (a).

(b)

$$T = \frac{l}{u_\text{p}} = \frac{200}{2\times10^8} = 1\ \mu\text{s}.$$

We will divide the problem into two parts, one for $V_{\text{g}_1}(t)$ and another for $V_{\text{g}_2}(t)$ and then we will use superposition to determine the solution for the sum. The solution for $V_{\text{g}_2}(t)$ will mimic the solution for $V_{\text{g}_1}(t)$, except for a reversal in sign and a delay by 0.4 μs.

For $V_{\text{g}_1}(t) = 200U(t)$:

$$\Gamma_\text{g} = \frac{R_\text{g}-Z_0}{R_\text{g}+Z_0} = \frac{25-75}{25+75} = -0.5,$$

$$\Gamma_\text{L} = \frac{Z_\text{L}-Z_0}{Z_\text{L}+Z_0} = \frac{125-75}{125+75} = 0.25,$$

$$V_1^+ = \frac{V_1 Z_0}{R_g + Z_0} = \frac{200 \times 75}{25 + 75} = 150 \text{ V},$$
$$V_\infty = \frac{V_g Z_L}{R_g + Z_L} = \frac{200 \times 125}{25 + 125} = 166.67 \text{ V}.$$

(i) $V_1(0,t)$ at sending end due to $V_{g_1}(t)$:

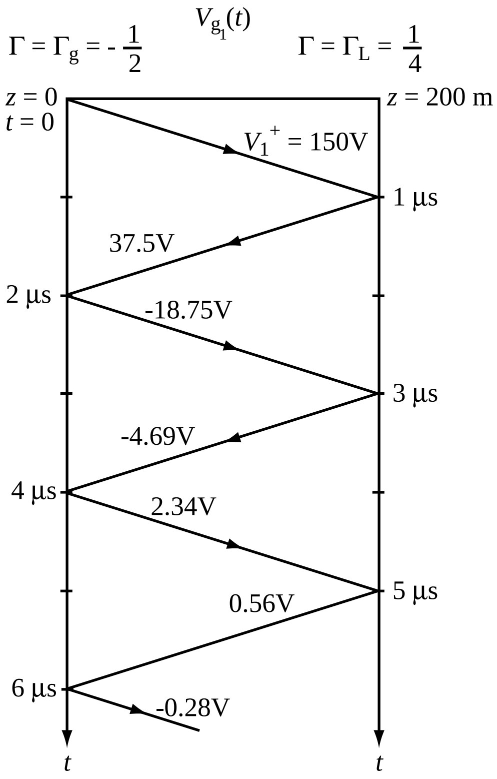


Figure P2.55: (c) Bounce diagram for voltage in reaction to $V_{g_1}(t)$.

(ii) $V_2(0,t)$ at sending end due to $V_{g_2}(t)$:

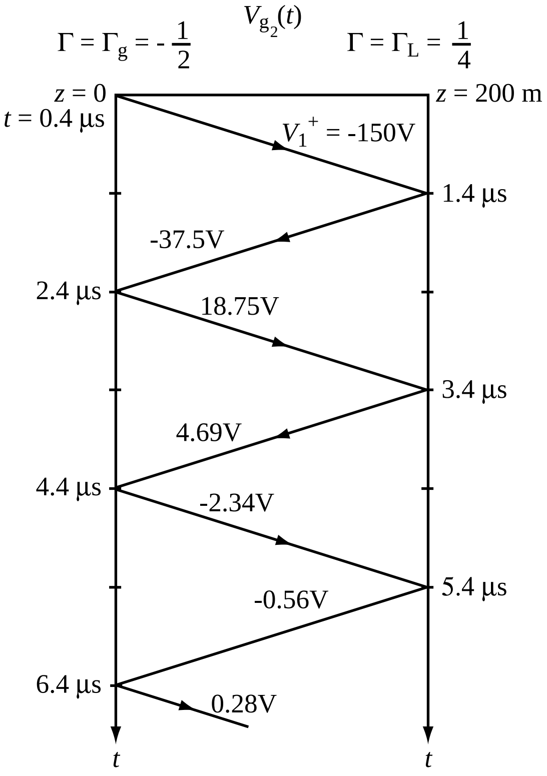


Figure P2.55: (d) Bounce diagram for voltage in reaction to $V_{g_2}(t)$.

(b)

(i) $V_1(0,t)$ at sending end due to $V_{g_1}(t)$:

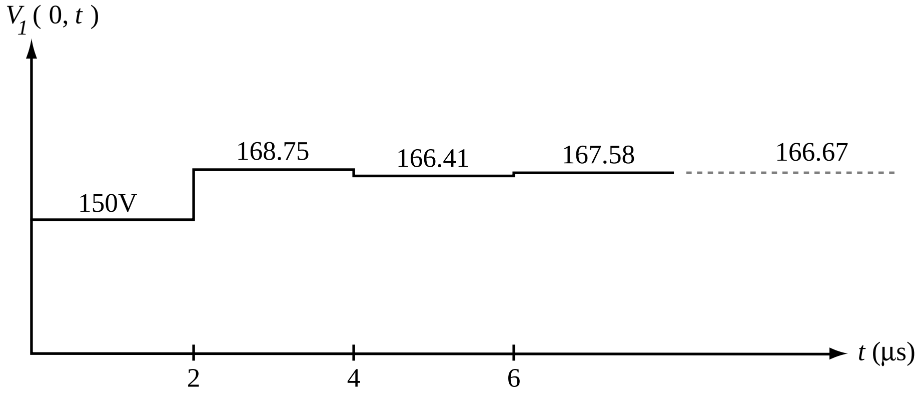


Figure P2.55: (e) $V_1(0,t)$.

(ii) $V_2(0,t)$ at sending end:

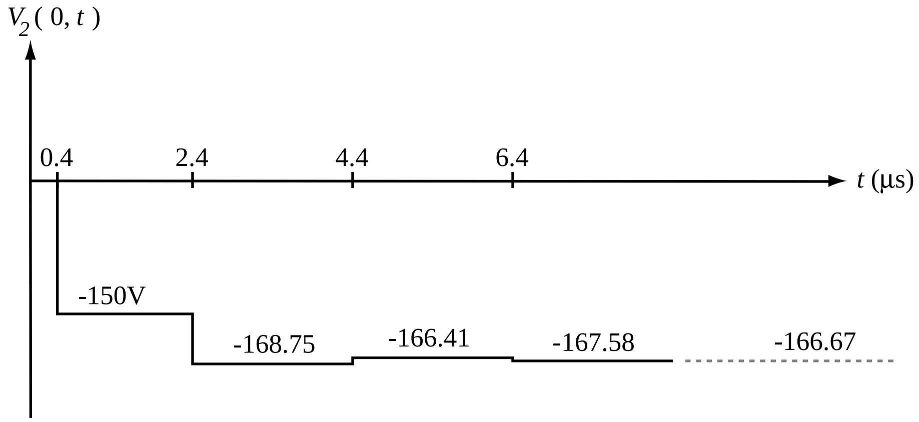


Figure P2.55: (f) $V_2(0,t)$.

(iii) Net voltage $V(0,t) = V_1(0,t) + V_2(0,t)$:

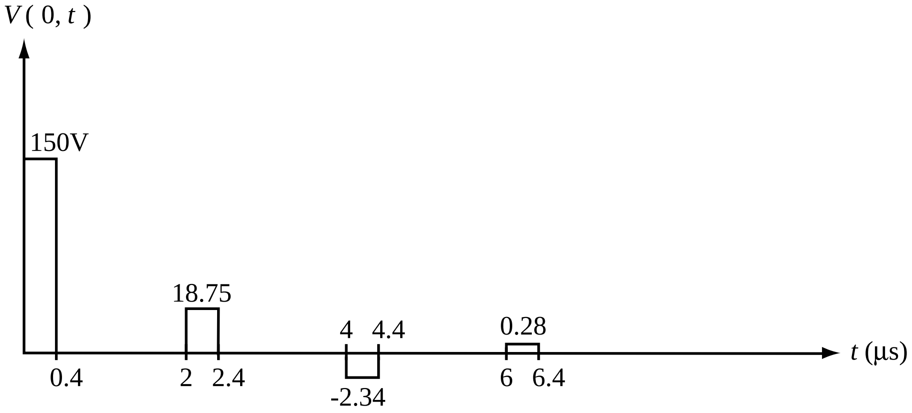


Figure P2.55: (g) Net voltage $V(0,t)$.

Problem 2.56 For the circuit of Problem 2.55, generate a bounce diagram for the current and plot its time history at the middle of the line.

Solution: Using the values for Γ_g and Γ_L calculated in Problem 2.55, we reverse their signs when using them to construct a bounce diagram for the current.

$$I_1^+ = \frac{V_1^+}{Z_0} = \frac{150}{75} = 2 \text{ A},$$
$$I_2^+ = \frac{V_2^+}{Z_0} = \frac{-150}{75} = -2 \text{ A},$$
$$I_\infty^+ = \frac{V_\infty}{Z_\text{L}} = 1.33 \text{ A}.$$

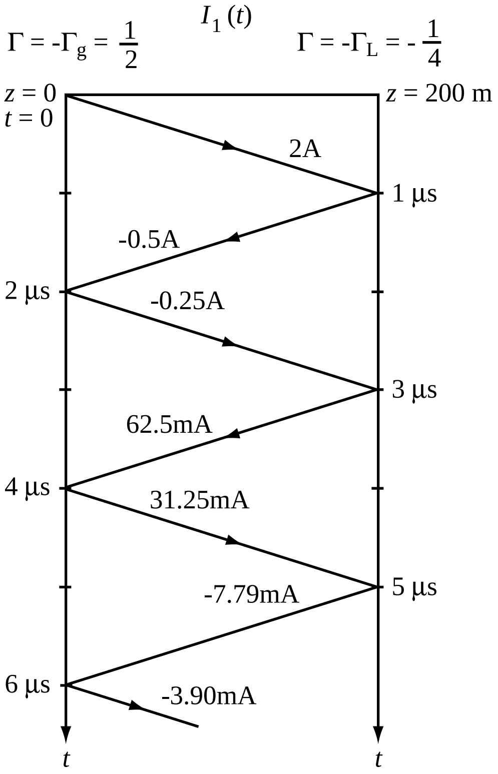


Figure P2.56: (a) Bounce diagram for $I_1(t)$ in reaction to $V_{g_1}(t)$.

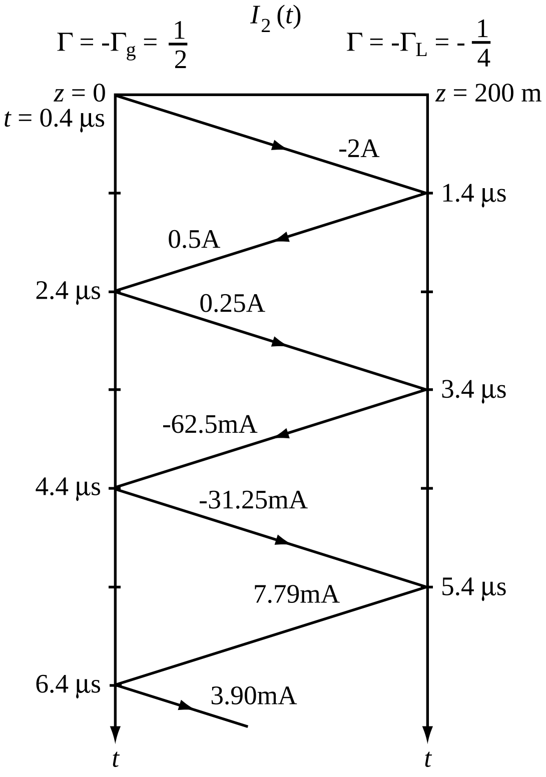


Figure P2.56: (b) Bounce diagram for current $I_2(t)$ in reaction to $V_{g_2}(t)$.

(i) $I_1(l/2,t)$ due to $V_{g_1}(t)$:

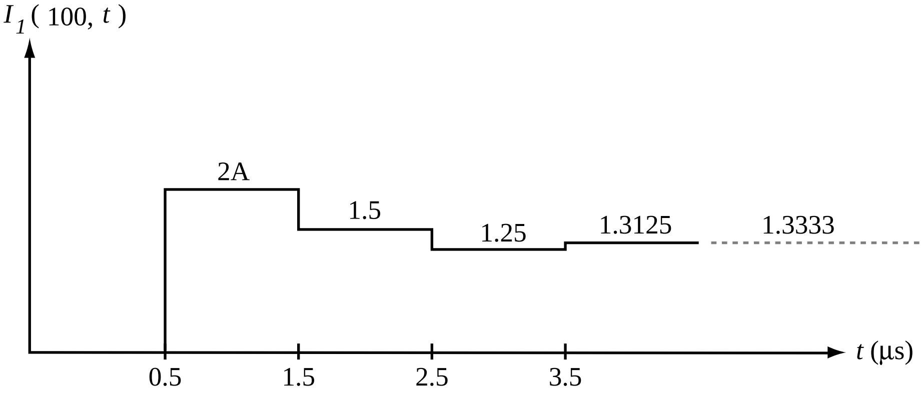


Figure P2.56: (c) $I_1(l/2,t)$.

(ii) $I_2(l/2,t)$ due to $V_{g_2}(t)$:

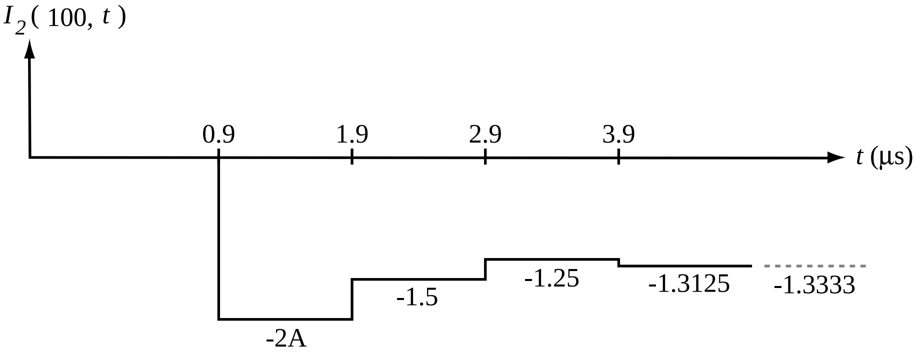


Figure P2.56: (d) $I_2(l/2,t)$.

(iii) Net current $I(l/2,t) = I_1(l/2,t) + I_2(l/2,t)$:

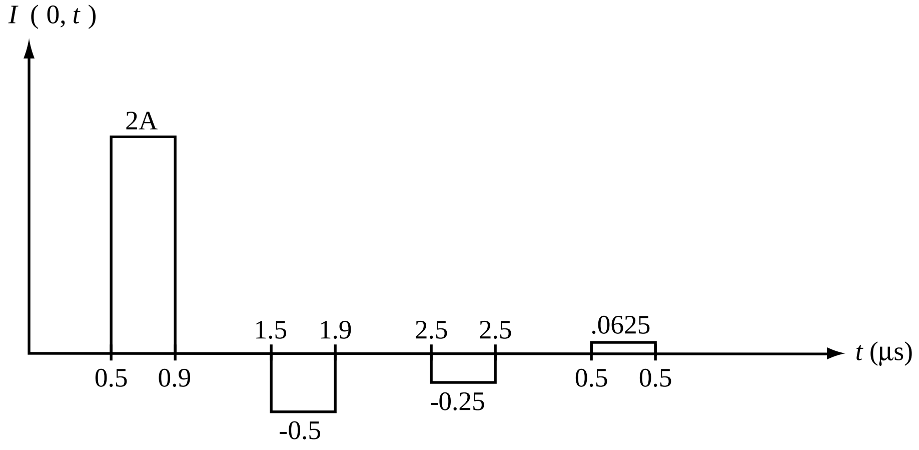


Figure P2.56: (e) Total $I(l/2,t)$.

Problem 2.57 For the parallel-plate transmission line of Problem 2.3, the line parameters are given by:

$$R' = 1 \quad (\Omega/\text{m}),$$
$$L' = 167 \quad (\text{nH/m}),$$
$$G' = 0,$$
$$C' = 172 \quad (\text{pF/m}).$$

Find α, β, u_p, and Z_0 at 1 GHz.

Solution: At 1 GHz, $\omega = 2\pi f = 2\pi \times 10^9$ rad/s. Application of (2.22) gives:

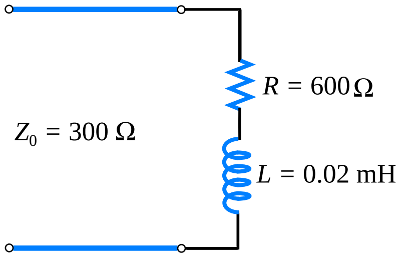
$$\begin{aligned}
\gamma &= \sqrt{(R' + j\omega L')(G' + j\omega C')} \\
&= [(1 + j2\pi \times 10^9 \times 167 \times 10^{-9})(0 + j2\pi \times 10^9 \times 172 \times 10^{-12})]^{1/2} \\
&= [(1 + j1049)(j1.1)]^{1/2} \\
&= \left[\sqrt{1 + (1049)^2}\, e^{j\tan^{-1} 1049} \times 1.1 e^{j90^\circ}\right]^{1/2}, \qquad (j = e^{j90^\circ}) \\
&= \left[1049 e^{j89.95^\circ} \times 1.1 e^{j90^\circ}\right]^{1/2} \\
&= \left[1154 e^{j179.95^\circ}\right]^{1/2} \\
&= 34 e^{j89.97^\circ} = 34 \cos 89.97^\circ + j34 \sin 89.97^\circ = 0.016 + j34.
\end{aligned}$$

Hence,

$$\begin{aligned}
\alpha &= 0.016 \text{ Np/m}, \\
\beta &= 34 \text{ rad/m}.
\end{aligned}$$

$$\begin{aligned}
u_{\text{p}} &= \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{2\pi \times 10^9}{34} = 1.85 \times 10^8 \text{ m/s}. \\
Z_0 &= \left[\frac{R' + j\omega L'}{G' + j\omega C'}\right]^{1/2} \\
&= \left[\frac{1049 e^{j89.95^\circ}}{1.1 e^{j90^\circ}}\right]^{1/2} \\
&= \left[954 e^{-j0.05^\circ}\right]^{1/2} \\
&= 31 e^{-j0.025^\circ} \simeq (31 - j0.01)\ \Omega.
\end{aligned}$$

Problem 2.58



A 300-Ω lossless air transmission line is connected to a complex load composed of a resistor in series with an inductor, as shown in the figure. At 5 MHz, determine: **(a)** Γ, **(b)** S, **(c)** location of voltage maximum nearest to the load, and **(d)** location of current maximum nearest to the load.

Solution:

(a)

$$\begin{aligned} Z_\mathrm{L} &= R + j\omega L \\ &= 600 + j2\pi \times 5 \times 10^6 \times 2 \times 10^{-5} = (600 + j628)\ \Omega. \end{aligned}$$

$$\begin{aligned} \Gamma &= \frac{Z_\mathrm{L} - Z_0}{Z_\mathrm{L} + Z_0} \\ &= \frac{600 + j628 - 300}{600 + j628 + 300} \\ &= \frac{300 + j628}{900 + j628} = 0.63e^{j29.6^\circ}. \end{aligned}$$

(b)

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.63}{1 - 0.63} = 1.67.$$

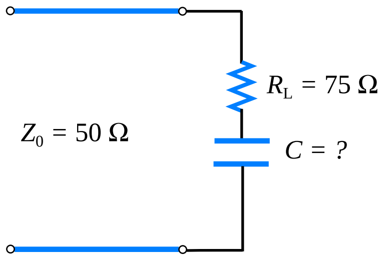
(c)

$$\begin{aligned} l_\mathrm{max} &= \frac{\theta_\mathrm{r}\lambda}{4\pi} \qquad \text{for } \theta_\mathrm{r} > 0. \\ &= \left(\frac{29.6^\circ \pi}{180^\circ}\right)\frac{60}{4\pi}, \qquad \left(\lambda = \frac{3 \times 10^8}{5 \times 10^6} = 60 \text{ m}\right) \\ &= 2.46 \text{ m} \end{aligned}$$

(d) The locations of current maxima correspond to voltage minima and vice versa. Hence, the location of current maximum nearest the load is the same as location of voltage minimum nearest the load. Thus

$$\begin{aligned} l_\mathrm{min} &= l_\mathrm{max} + \frac{\lambda}{4}, \qquad \left(l_\mathrm{max} < \frac{\lambda}{4} = 15 \text{ m}\right) \\ &= 2.46 + 15 = 17.46 \text{ m}. \end{aligned}$$

Problem 2.59



A 50-Ω lossless transmission line is connected to a load composed of a 75-Ω resistor in series with a capacitor of unknown capacitance. If at 10 MHz the voltage standing wave ratio on the line was measured to be 3, determine the capacitance C.

Solution:

$$|\Gamma| = \frac{S-1}{S+1} = \frac{3-1}{3+1} = \frac{2}{4} = 0.5$$

$$Z_L = R_L - jX_C, \qquad \text{where } X_C = \frac{1}{\omega C}\,.$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$|\Gamma|^2 = \left[\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)\left(\frac{Z_L^* - Z_0}{Z_L^* + Z_0}\right)\right]$$

$$|\Gamma|^2 = \frac{Z_L Z_L^* + Z_0^2 - Z_0(Z_L + Z_L^*)}{Z_L Z_L^* + Z_0^2 + Z_0(Z_L + Z_L^*)}$$

Noting that:

$$Z_L Z_L^* = (R_L - jX_C)(R_L + jX_C) = R_L^2 + X_C^2,$$

$$Z_0(Z_L + Z_L^*) = Z_0(R_L - jX_C + R_L + jX_C) = 2Z_0R_L,$$

$$|\Gamma|^2 = \frac{R_L^2 + X_C^2 + Z_0^2 - 2Z_0R_L}{R_L^2 + X_C^2 + Z_0^2 + 2Z_0R_L}\,.$$

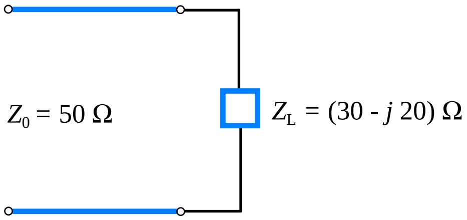
Upon substituting $|\Gamma_L| = 0.5$, $R_L = 75$ Ω, and $Z_0 = 50$ Ω, and then solving for X_C, we have

$$X_C = 66.1 \ \Omega.$$

Hence

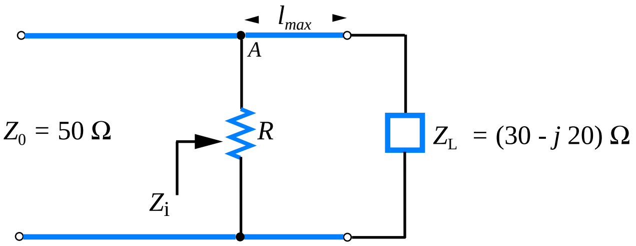
$$C = \frac{1}{\omega X_C} = \frac{1}{2\pi \times 10^7 \times 66.1} = 2.41 \times 10^{-10} = 241 \text{ pF}.$$

Problem 2.60 A 50-Ω lossless line is terminated in a load impedance $Z_L = (30 - j20)$ Ω.



(a) Calculate Γ and S.

(b) It has been proposed that by placing an appropriately selected resistor across the line at a distance $l_{\max}$ from the load (as shown in the figure below), where $l_{\max}$ is the distance from the load of a voltage maximum, then it is possible to render $Z_i = Z_0$, thereby eliminating reflections back to the sending end. Show that the proposed approach is valid and find the value of the shunt resistance.



Solution:

(a)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{30 - j20 - 50}{30 - j20 + 50} = \frac{-20 - j20}{80 - j20} = \frac{-(20 + j20)}{80 - j20} = 0.34e^{-j121^\circ}.$$

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.34}{1 - 0.34} = 2.$$

(b) We start by finding $l_{\max}$, the distance of the voltage maximum nearest to the load. Using (2.56) with $n = 1$,

$$\begin{aligned} l_{\max} &= \frac{\theta_r \lambda}{4\pi} + \frac{\lambda}{2} \\ &= \left(\frac{-121^\circ \pi}{180^\circ}\right)\frac{\lambda}{4\pi} + \frac{\lambda}{2} = 0.33\lambda. \end{aligned}$$

Applying (2.63) at $l = l_{\max} = 0.33\lambda$, for which $\beta l = (2\pi/\lambda \times 0.33\lambda = 2.07$ radians, the value of Z_{in} before adding the shunt resistance is:

$$\begin{aligned} Z_{\text{in}} &= Z_0 \left(\frac{Z_{\text{L}} + jZ_0 \tan\beta l}{Z_0 + jZ_{\text{L}} \tan\beta l} \right) \\ &= 50 \left(\frac{(30 - j20) + j50 \tan 2.07}{50 + j(30 - j20) \tan 2.07} \right) = (102 + j0)\ \Omega. \end{aligned}$$

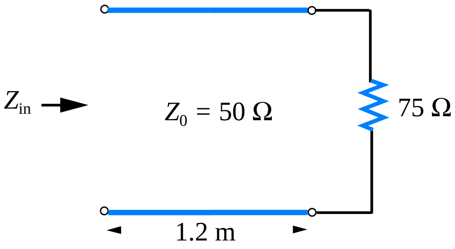
Thus, at the location A (at a distance l_{max} from the load), the input impedance is purely real. If we add a shunt resistor R in parallel such that the combination is equal to Z_0, then the new Z_{in} at any point to the left of that location will be equal to Z_0.

Hence, we need to select R such that

$$\frac{1}{R} + \frac{1}{102} = \frac{1}{50}$$

or $R = 98\ \Omega$.

Problem 2.61 For the lossless transmission line circuit shown in the figure, determine the equivalent series lumped-element circuit at 400 MHz at the input to the line. The line has a characteristic impedance of 50 Ω and the insulating layer has $\varepsilon_r = 2.25$.



Solution: At 400 MHz,

$$\begin{aligned} \lambda &= \frac{u_{\text{p}}}{f} = \frac{c}{f\sqrt{\varepsilon_{\text{r}}}} = \frac{3 \times 10^8}{4 \times 10^8 \sqrt{2.25}} = 0.5 \text{ m}. \\ \beta l &= \frac{2\pi}{\lambda} l = \frac{2\pi}{0.5} \times 1.2 = 4.8\pi. \end{aligned}$$

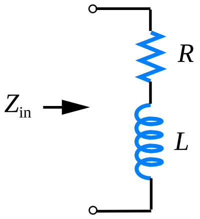
Subtracting multiples of 2π, the remainder is:

$$\beta l = 0.8\pi \text{ rad}.$$

Using (2.63),

$$Z_{\text{in}} = Z_0 \left(\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \right)$$
$$= 50 \left(\frac{75 + j50 \tan 0.8\pi}{50 + j75 \tan 0.8\pi} \right) = (52.38 + j20.75)\ \Omega.$$

Z_{in} is equivalent to a series RL circuit with



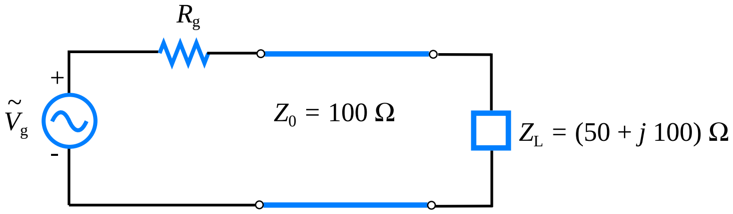
$$R = 52.38\ \Omega$$
$$\omega L = 2\pi f L = 20.75\ \Omega$$

or

$$L = \frac{20.75}{2\pi \times 4 \times 10^8} = 8.3 \times 10^{-9}\ \text{H},$$

which is a very small inductor.

Problem 2.62



The circuit shown in the figure consists of a 100-Ω lossless transmission line terminated in a load with $Z_L = (50 + j100)\ \Omega$. If the peak value of the load voltage was measured to be $|\widetilde{V}_L| = 12$ V, determine:

- **(a)** the time-average power dissipated in the load,
- **(b)** the time-average power incident on the line, and
- **(c)** the time-average power reflected by the load.

Solution:

(a)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 + j100 - 100}{50 + j100 + 100} = \frac{-50 + j100}{150 + j100} = 0.62e^{j82.9^\circ}.$$

The time average power dissipated in the load is:

$$\begin{aligned} P_{av} &= \frac{1}{2}|\widetilde{I}_L|^2 R_L \\ &= \frac{1}{2}\left|\frac{\widetilde{V}_L}{Z_L}\right|^2 R_L \\ &= \frac{1}{2}\frac{|\widetilde{V}_L|^2}{|Z_L|^2} R_L = \frac{1}{2} \times 12^2 \times \frac{50}{50^2 + 100^2} = 0.29 \text{ W.} \end{aligned}$$

(b)

$$P_{av} = P^i_{av}(1 - |\Gamma|^2)$$

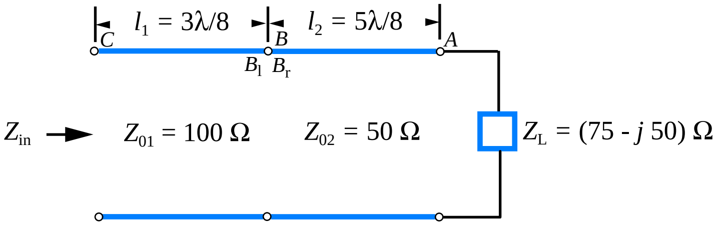
Hence,

$$P^i_{av} = \frac{P_{av}}{1 - |\Gamma|^2} = \frac{0.29}{1 - 0.62^2} = 0.47 \text{ W.}$$

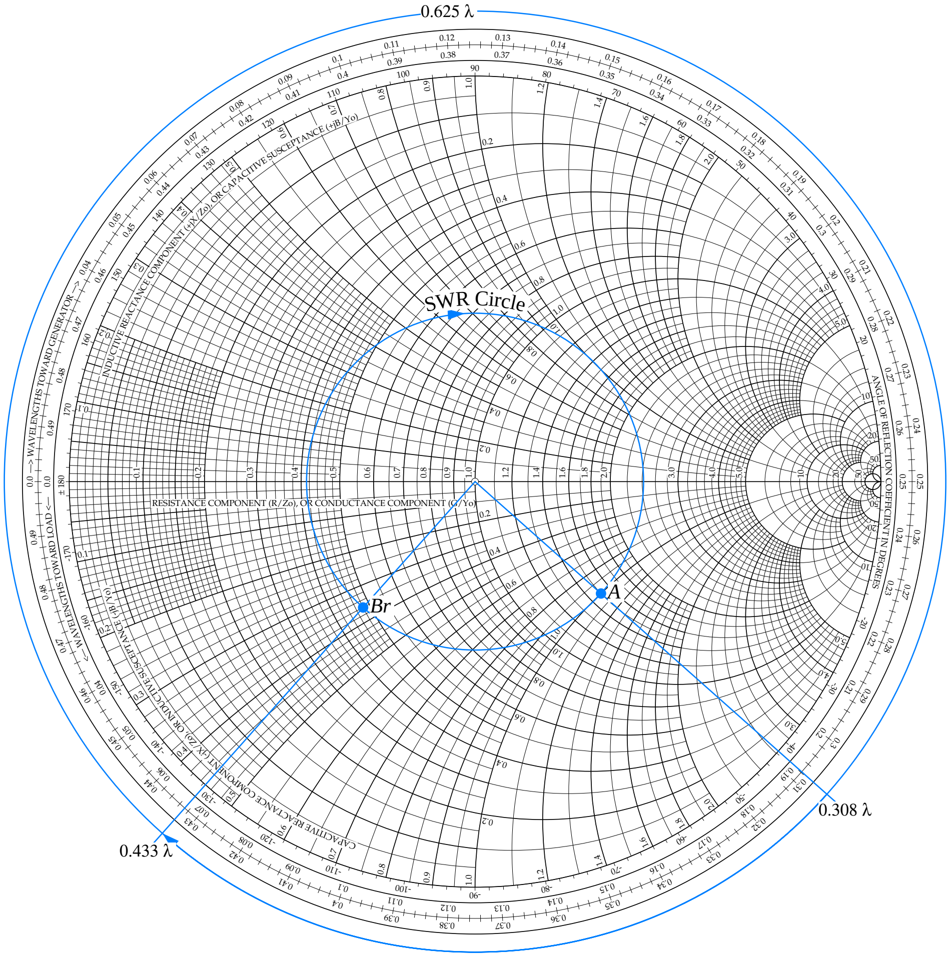
(c)

$$P^r_{av} = -|\Gamma|^2 P^i_{av} = -(0.62)^2 \times 0.47 = -0.18 \text{ W.}$$

Problem 2.63



Use the Smith chart to determine the input impedance Z_{in} of the two-line configuration shown in the figure.


0.625 λ
SWR Circle
A
Br
0.308 λ
0.433 λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
0.0 —> WAVELENGTHS TOWARD GENERATOR —>
<— WAVELENGTHS TOWARD LOAD <—
ANGLE OF REFLECTION COEFFICIENT IN DEGREES

Smith Chart 1

Solution: Starting at point A, namely at the load, we normalize Z_L with respect to Z_{02}:

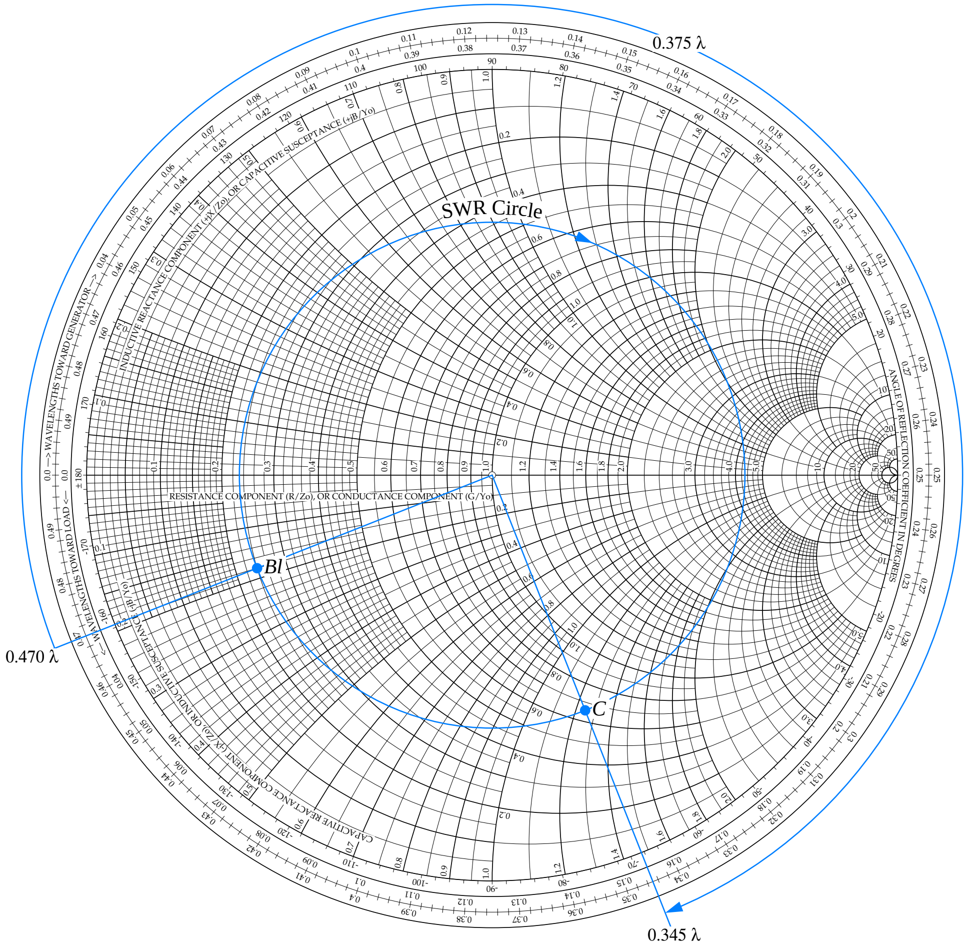
$$z_\mathrm{L} = \frac{Z_\mathrm{L}}{Z_{02}} = \frac{75 - j50}{50} = 1.5 - j1. \qquad \text{(point } A \text{ on Smith chart 1)}$$

From point A on the Smith chart, we move on the SWR circle a distance of $5\lambda/8$ to point B_r, which is just to the right of point B (see figure). At B_r, the normalized input impedance of line 2 is:

$$z_\mathrm{in2} = 0.48 - j0.36 \qquad \text{(point } B_\mathrm{r} \text{ on Smith chart)}$$

Next, we unnormalize z_in2:

$$Z_\mathrm{in2} = Z_{02} z_\mathrm{in2} = 50 \times (0.48 - j0.36) = (24 - j18)\ \Omega.$$



Smith Chart 2

To move along line 1, we need to normalize with respect to Z_{01}. We shall call this z_{L1}:

$$z_{L1} = \frac{Z_{in2}}{Z_{01}} = \frac{24 - j18}{100} = 0.24 - j0.18 \qquad \text{(point } B_\ell \text{ on Smith chart 2)}$$

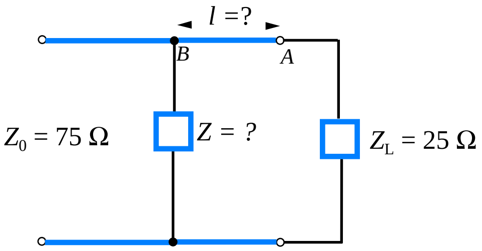
After drawing the SWR circle through point B_ℓ, we move $3\lambda/8$ towards the generator, ending up at point C on Smith chart 2. The normalized input impedance of line 1 is:

$$z_{in} = 0.66 - j1.25$$

which upon unnormalizing becomes:

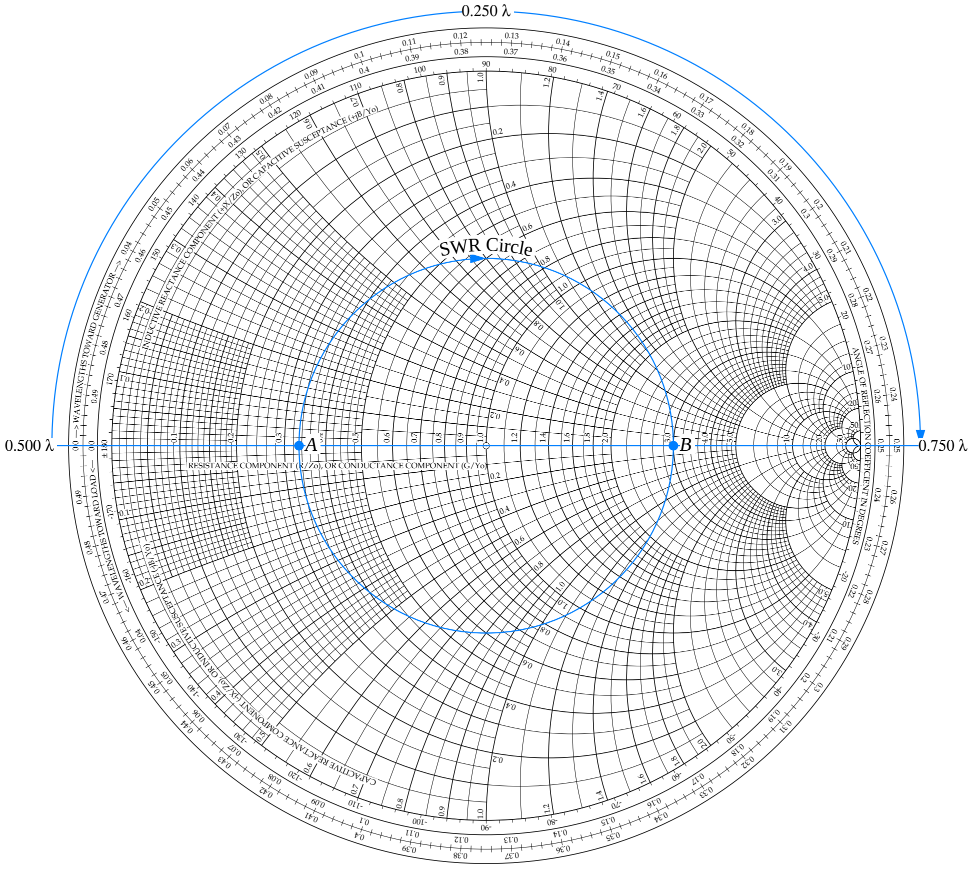
$$Z_{in} = (66 - j125)\ \Omega.$$

Problem 2.64



A 25-Ω antenna is connected to a 75-Ω lossless transmission line. Reflections back toward the generator can be eliminated by placing a shunt impedance Z at a distance l from the load. Determine the values of Z and l.

Solution:



The normalized load impedance is:

$$z_L = \frac{25}{75} = 0.33 \qquad \text{(point } A \text{ on Smith chart)}$$

The Smith chart shows A and the SWR circle. The goal is to have an equivalent impedance of 75 Ω to the left of B. That equivalent impedance is the parallel combination of Z_{in} at B (to the right of the shunt impedance Z) and the shunt element Z. Since we need for this to be purely real, it's best to choose l such that Z_{in} is purely real, thereby choosing Z to be simply a resistor. Adding two resistors in parallel generates a sum smaller in magnitude than either one of them. So we need for Z_{in} to be larger than Z_0, not smaller. On the Smith chart, that point is B, at a distance $l = \lambda/4$ from the load. At that point:

$$z_{in} = 3,$$

which corresponds to

$$y_{in} = 0.33.$$

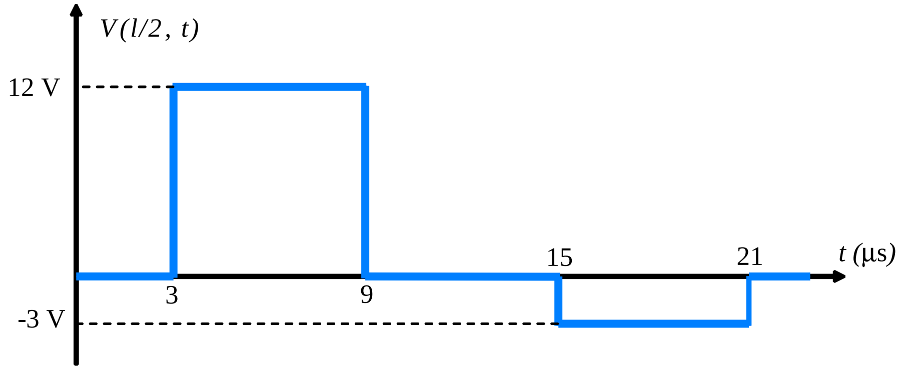
Hence, we need y, the normalized admittance corresponding to the shunt impedance Z, to have a value that satisfies:

$$\begin{aligned} y_{in} + y &= 1 \\ y &= 1 - y_{in} = 1 - 0.33 = 0.66 \\ z &= \frac{1}{y} = \frac{1}{0.66} = 1.5 \\ Z &= 75 \times 1.5 = 112.5\ \Omega. \end{aligned}$$

In summary,

$$\begin{aligned} l &= \frac{\lambda}{4}, \\ Z &= 112.5\ \Omega. \end{aligned}$$

Problem 2.65 In response to a step voltage, the voltage waveform shown in the figure below was observed at the midpoint of a lossless transmission line with $Z_0 = 50\ \Omega$ and $u_p = 2 \times 10^8$ m/s. Determine: **(a)** the length of the line, **(b)** Z_L, **(c)** R_g, and **(d)** V_g.



Solution:

(a) Since it takes 3 μs to reach the middle of the line, the line length must be

$$l = 2(3 \times 10^{-6} \times u_p) = 2 \times 3 \times 10^{-6} \times 2 \times 10^{8} = 1200 \text{ m}.$$

(b) From the voltage waveform shown in the figure, the duration of the first rectangle is 6 μs, representing the time it takes the incident voltage V_1^+ to travel from the midpoint of the line to the load and back. The fact that the voltage drops to zero at $t = 9$ μs implies that the reflected wave is exactly equal to V_1^+ in magnitude, but opposite in polarity. That is,

$$V_1^- = -V_1^+.$$

This in turn implies that $\Gamma_L = -1$, which means that the load is a short circuit:

$$Z_L = 0.$$

(c) After V_1^- arrives at the generator end, it encounters a reflection coefficient Γ_g. The voltage at 15 μs is composed of:

$$\begin{aligned} V &= V_1^+ + V_1^- + V_2^+ \\ &= (1 + \Gamma_L + \Gamma_L\Gamma_g)V_1^+ \\ \frac{V}{V_1^+} &= 1 - 1 - \Gamma_g \end{aligned}$$

From the figure, $V/V_1^+ = -3/12 = -1/4$. Hence,

$$\Gamma_g = \frac{1}{4},$$

which means that

$$R_g = \left(\frac{1+\Gamma_g}{1-\Gamma_g}\right) Z_0 = \left(\frac{1+0.25}{1-0.25}\right) 50 = 83.3\ \Omega.$$

(d)

$$V_1^+ = 12 = \frac{V_g Z_0}{R_g + Z_0}$$

$$V_g = \frac{12(R_g + Z_0)}{Z_0} = \frac{12(83.3+50)}{50} = 32\ \text{V}.$$

Chapter 3: Vector Analysis

Lesson #14

Chapter — Section: 3-1

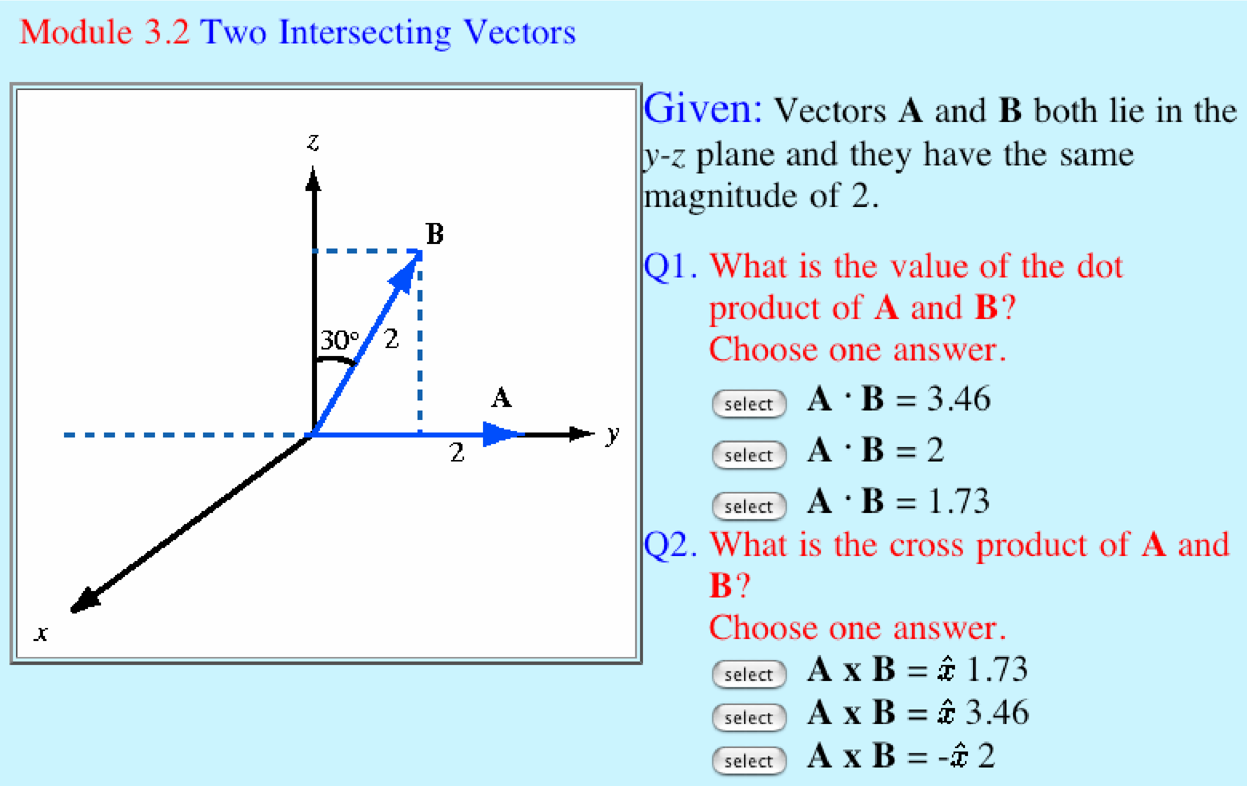
Topics: Basic laws of vector algebra

Highlights:

- Vector magnitude, direction, unit vector
- Position and distance vectors
- Vector addition and multiplication
    - Dot product
    - Vector product
    - Triple product

Special Illustrations:

- CD-ROM Module 3.2



Module 3.2 Two Intersecting Vectors

Given: Vectors **A** and **B** both lie in the y-z plane and they have the same magnitude of 2.

Q1. What is the value of the dot product of **A** and **B**?
Choose one answer.

- $\mathbf{A} \cdot \mathbf{B} = 3.46$
- $\mathbf{A} \cdot \mathbf{B} = 2$
- $\mathbf{A} \cdot \mathbf{B} = 1.73$

Q2. What is the cross product of **A** and **B**?
Choose one answer.

- $\mathbf{A} \times \mathbf{B} = \hat{x}\, 1.73$
- $\mathbf{A} \times \mathbf{B} = \hat{x}\, 3.46$
- $\mathbf{A} \times \mathbf{B} = -\hat{x}\, 2$

Lessons #15 and 16

Chapter — Section: 3-2

Topics: Coordinate systems

Highlights:

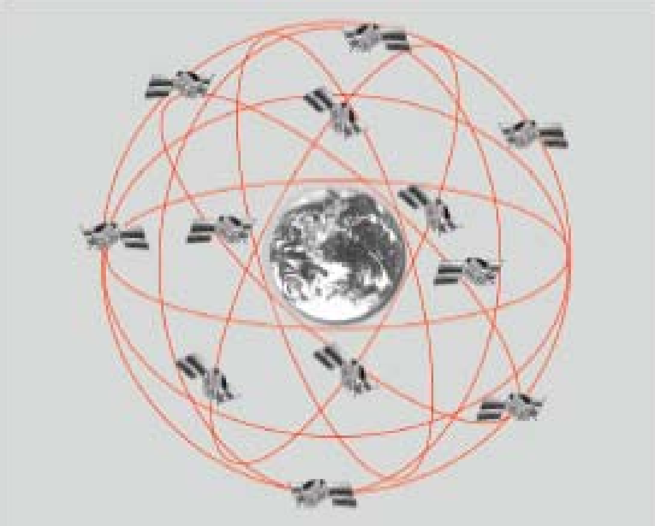
- Commonly used coordinate systems: Cartesian, cylindrical, spherical
- Choice is based on which one best suits problem geometry
- Differential surface vectors and differential volumes

Special Illustrations:

- Examples 3-3 to 3-5
- Technology Brief on “GPS” (CD-ROM)

Global Positioning System

The Global Positioning System (GPS), initially developed in the 1980s by the U.S. Department of Defense as a navigation tool for military use, has evolved into a system with numerous civilian applications including vehicle tracking, aircraft navigation, map displays in automobiles, and topographic mapping. The overall GPS is composed of 3 segments. The space segment consists of 24 satellites (A), each circling Earth every 12 hours at an orbital altitude of about 12,000 miles and transmitting continuous coded time signals. The user segment consists of hand-held or vehicle-mounted receivers that determine their own locations by receiving and processing multiple satellite signals. The third segment is a network of five ground stations, distributed around the world, that monitor the satellites and provide them with updates on their precise orbital information. GPS provides a location inaccuracy of about 30 m, both horizontally and vertically, but it can be improved to within 1 m by differential GPS (see illustration).



A. GPS nominal satellite constellation
4 Satellites in each Plane
20,200 km Altitudes, 55 Degree Inclination

Lesson #17

Chapter — Section: 3-3

Topics: Coordinate transformations

Highlights:

- Basic logic for decomposing a vector in one coordinate system into the coordinate variables of another system
- Transformation relations (Table 3-2)

Special Illustrations:

- Example 3-8

Lesson #18

Chapter — Section: 3-4

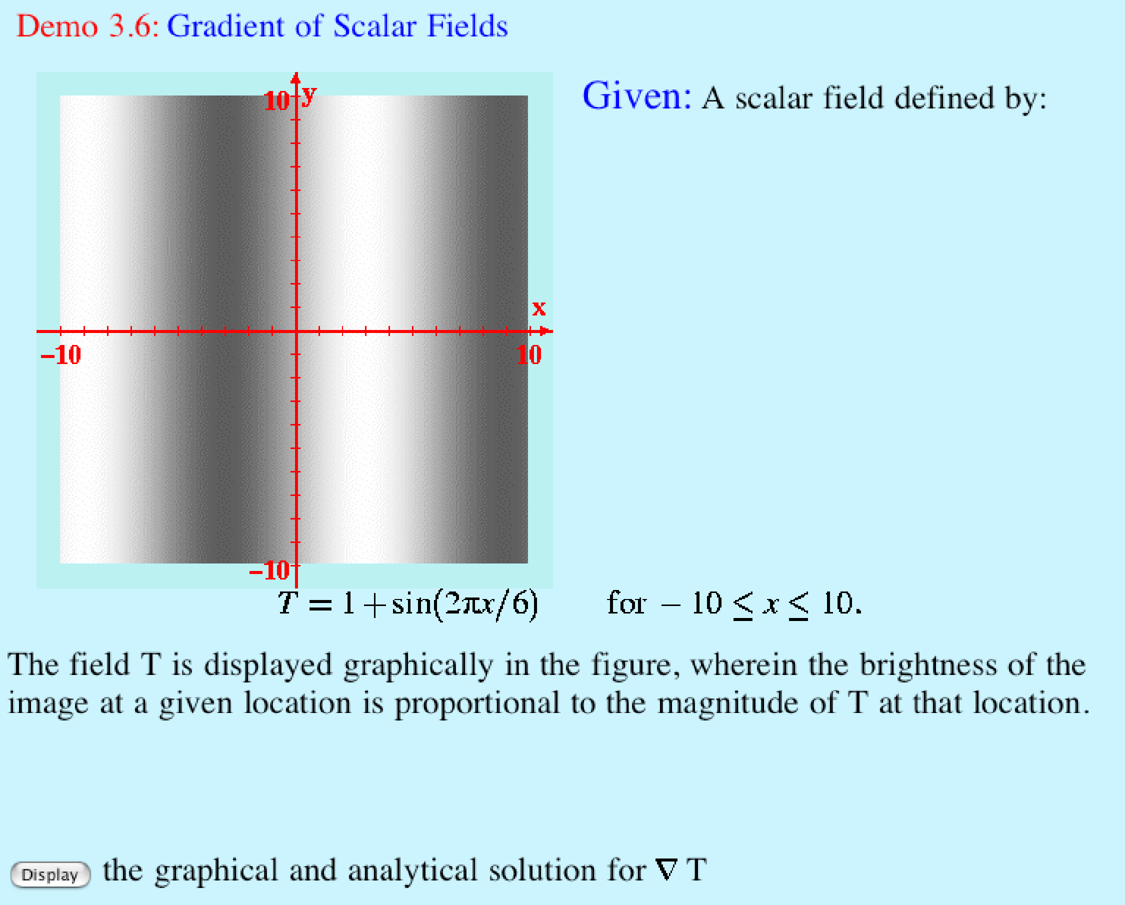
Topics: Gradient operator

Highlights:

- Derivation of ∇T in Cartesian coordinates
- Directional derivative
- ∇T in cylindrical and spherical coordinates

Special Illustrations:

- Example 3-10(b)
- CD-ROM Modules 3.5 or 3.6
- CD-ROM Demos 3.1-3.9 (any 2)



Demo 3.6: Gradient of Scalar Fields

Given: A scalar field defined by:

$T = 1 + \sin(2\pi x/6)$ for $-10 \le x \le 10$.

The field T is displayed graphically in the figure, wherein the brightness of the image at a given location is proportional to the magnitude of T at that location.

Display the graphical and analytical solution for ∇ T

Lesson #19

Chapter — Section: 3-5

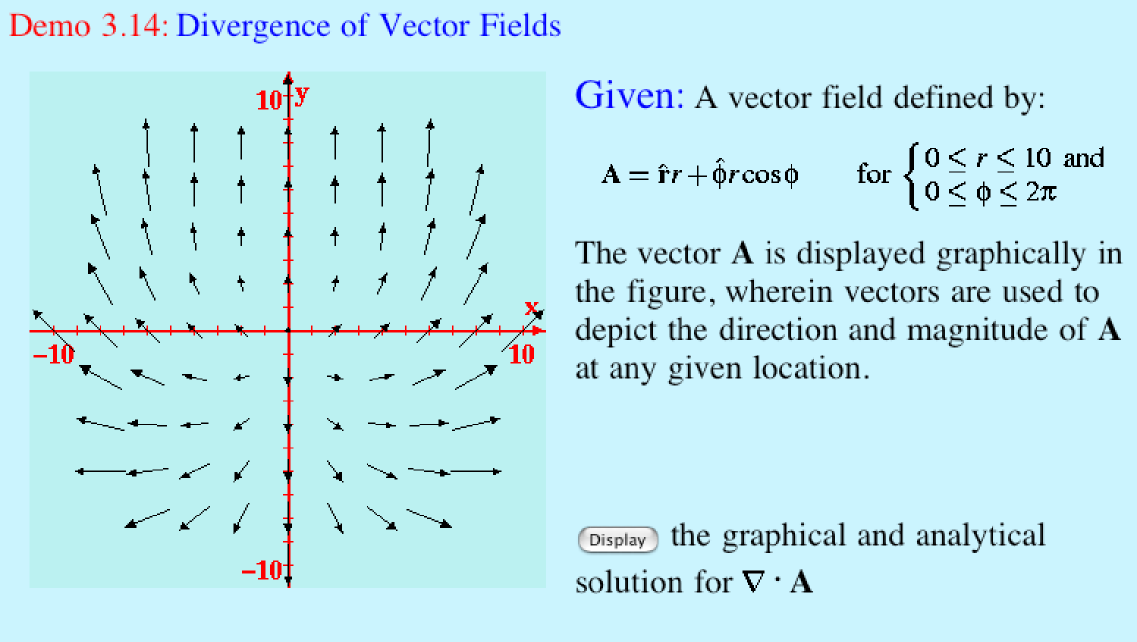
Topics: Divergence operator

Highlights:

- Concept of "flux"
- Derivation of ∇ .**E**
- Divergence theorem

Special Illustrations:

- CD-ROM Modules 3.7-3.11 (any 2)
- CD-ROM Demos 3.10-3.15 (any 1 or 2)



Demo 3.14: Divergence of Vector Fields

Given: A vector field defined by:

$$\mathbf{A} = \hat{\mathbf{r}}\, r + \hat{\boldsymbol{\phi}}\, r\cos\phi \qquad \text{for} \begin{cases} 0 \le r \le 10 \text{ and} \\ 0 \le \phi \le 2\pi \end{cases}$$

The vector **A** is displayed graphically in the figure, wherein vectors are used to depict the direction and magnitude of **A** at any given location.

the graphical and analytical solution for $\nabla \cdot \mathbf{A}$

Lesson #20

Chapter — Section: 3-6

Topics: Curl operator

Highlights:

- Concept of “circulation”
- Derivation of $\nabla \times \mathbf{B}$
- Stokes’s theorem

Special Illustrations:

- Example 3-12

Lesson #21

Chapter — Section: 3-7

Topics: Laplacian operator

Highlights:

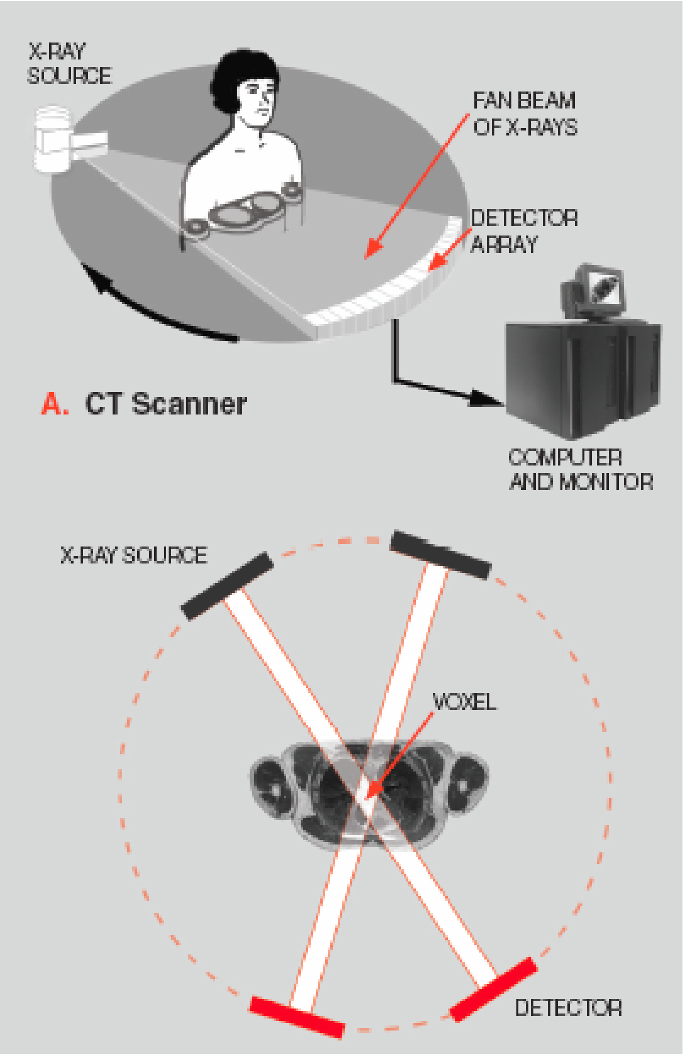
- Definition of $\nabla^2 V$
- Definition of $\nabla^2 \mathbf{E}$

Special Illustrations:

- Technology Brief on "X-Ray Computed Tomography"

X-Ray Computed Tomography

Tomography is derived from the Greek words *tome*, meaning section or slice, and *graphia*, meaning writing. Computed tomography, also known as CT scan or CAT scan (for computed axial tomography), refers to a technique capable of generating 3-D images of the x-ray attenuation (absorption) properties of an object. This is in contrast with the traditional x-ray technique which produces only a 2-D profile of the object. CT was invented in 1972 by British electrical engineer Godfrey Hounsfield, and independently by Allan Cormack, a South African-born American physicist. The two inventors shared the 1979 Nobel Prize for Physiology or Medicine. Among diagnostic imaging techniques, CT has the decided advantage in having the sensitivity to image body parts on a wide range of densities, from soft tissue to blood vessels and bones.



Chapter 3

Section 3-1: Vector Algebra

Problem 3.1 Vector **A** starts at point $(1,-1,-3)$ and ends at point $(2,-1,0)$. Find a unit vector in the direction of **A**.

Solution:

$$\begin{aligned}
\mathbf{A} &= \hat{\mathbf{x}}(2-1)+\hat{\mathbf{y}}(-1-(-1))+\hat{\mathbf{z}}(0-(-3)) = \hat{\mathbf{x}}+\hat{\mathbf{z}}3, \\
|\mathbf{A}| &= \sqrt{1+9} = 3.16, \\
\hat{\mathbf{a}} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}}+\hat{\mathbf{z}}3}{3.16} = \hat{\mathbf{x}}0.32+\hat{\mathbf{z}}0.95.
\end{aligned}$$

Problem 3.2 Given vectors $\mathbf{A} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}}$, $\mathbf{B} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}+\hat{\mathbf{z}}3$, and $\mathbf{C} = \hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2$, show that **C** is perpendicular to both **A** and **B**.

Solution:

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{C} &= (\hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}})\cdot(\hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2) = 8-6-2 = 0, \\
\mathbf{B}\cdot\mathbf{C} &= (\hat{\mathbf{x}}2-\hat{\mathbf{y}}+\hat{\mathbf{z}}3)\cdot(\hat{\mathbf{x}}4+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2) = 8-2-6 = 0.
\end{aligned}$$

Problem 3.3 In Cartesian coordinates, the three corners of a triangle are $P_1(0,4,4)$, $P_2(4,-4,4)$, and $P_3(2,2,-4)$. Find the area of the triangle.

Solution: Let $\mathbf{B} = \overrightarrow{P_1P_2} = \hat{\mathbf{x}}4-\hat{\mathbf{y}}8$ and $\mathbf{C} = \overrightarrow{P_1P_3} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}2-\hat{\mathbf{z}}8$ represent two sides of the triangle. Since the magnitude of the cross product is the area of the parallelogram (see the definition of cross product in Section 3-1.4), half of this is the area of the triangle:

$$\begin{aligned}
A = \tfrac{1}{2}|\mathbf{B}\times\mathbf{C}| &= \tfrac{1}{2}|(\hat{\mathbf{x}}4-\hat{\mathbf{y}}8)\times(\hat{\mathbf{x}}2-\hat{\mathbf{y}}2-\hat{\mathbf{z}}8)| \\
&= \tfrac{1}{2}|\hat{\mathbf{x}}(-8)(-8)+\hat{\mathbf{y}}(-(4)(-8))+\hat{\mathbf{z}}(4(-2)-(-8)2)| \\
&= \tfrac{1}{2}|\hat{\mathbf{x}}64+\hat{\mathbf{y}}32+\hat{\mathbf{z}}8| = \tfrac{1}{2}\sqrt{64^2+32^2+8^2} = \tfrac{1}{2}\sqrt{5184} = 36,
\end{aligned}$$

where the cross product is evaluated with Eq. (3.27).

Problem 3.4 Given $\mathbf{A} = \hat{\mathbf{x}}2-\hat{\mathbf{y}}3+\hat{\mathbf{z}}1$ and $\mathbf{B} = \hat{\mathbf{x}}B_x+\hat{\mathbf{y}}2+\hat{\mathbf{z}}B_z$:

(a) find B_x and B_z if **A** is parallel to **B**;

(b) find a relation between B_x and B_z if **A** is perpendicular to **B**.

Solution:

(a) If **A** is parallel to **B**, then their directions are equal or opposite: $\hat{\mathbf{a}}_A = \pm\hat{\mathbf{a}}_B$, or

$$\begin{aligned} \mathbf{A}/|\mathbf{A}| &= \pm\mathbf{B}/|\mathbf{B}|, \\ \frac{\hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}}{\sqrt{14}} &= \pm\frac{\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}B_z}{\sqrt{4 + B_x^2 + B_z^2}}\,. \end{aligned}$$

From the y-component,

$$\frac{-3}{\sqrt{14}} = \frac{\pm 2}{\sqrt{4 + B_x^2 + B_z^2}}$$

which can only be solved for the minus sign (which means that **A** and **B** must point in opposite directions for them to be parallel). Solving for $B_x^2 + B_z^2$,

$$B_x^2 + B_z^2 = \left(\frac{-2}{-3}\sqrt{14}\right)^2 - 4 = \frac{20}{9}\,.$$

From the x-component,

$$\frac{2}{\sqrt{14}} = \frac{-B_x}{\sqrt{56/9}}\,, \qquad B_x = \frac{-2\sqrt{56}}{3\sqrt{14}} = \frac{-4}{3}$$

and, from the z-component,

$$B_z = \frac{-2}{3}\,.$$

This is consistent with our result for $B_x^2 + B_z^2$.

These results could also have been obtained by assuming θ_{AB} was 0° or 180° and solving $|\mathbf{A}||\mathbf{B}| = \pm\mathbf{A}\cdot\mathbf{B}$, or by solving $\mathbf{A}\times\mathbf{B} = 0$.

(b) If **A** is perpendicular to **B**, then their dot product is zero (see Section 3-1.4). Using Eq. (3.17),

$$0 = \mathbf{A}\cdot\mathbf{B} = 2B_x - 6 + B_z,$$

or

$$B_z = 6 - 2B_x.$$

There are an infinite number of vectors which could be **B** and be perpendicular to **A**, but their x- and z-components must satisfy this relation.

This result could have also been obtained by assuming $\theta_{AB} = 90°$ and calculating $|\mathbf{A}||\mathbf{B}| = |\mathbf{A}\times\mathbf{B}|$.

Problem 3.5 Given vectors $\mathbf{A} = \hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3$, $\mathbf{B} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}4$, and $\mathbf{C} = \hat{\mathbf{y}}2 - \hat{\mathbf{z}}4$, find

(a) A and $\hat{\mathbf{a}}$,
(b) the component of **B** along **C**,
(c) θ_{AC},
(d) $\mathbf{A}\times\mathbf{C}$,
(e) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$,
(f) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$,
(g) $\hat{\mathbf{x}}\times\mathbf{B}$, and
(h) $(\mathbf{A}\times\hat{\mathbf{y}})\cdot\hat{\mathbf{z}}$.

Solution:

(a) From Eq. (3.4),

$$A=\sqrt{1^2+2^2+(-3)^2}=\sqrt{14},$$

and, from Eq. (3.5),

$$\hat{\mathbf{a}}_A=\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}2-\hat{\mathbf{z}}3}{\sqrt{14}}.$$

(b) The component of **B** along **C** (see Section 3-1.4) is given by

$$B\cos\theta_{BC}=\frac{\mathbf{B}\cdot\mathbf{C}}{C}=\frac{-8}{\sqrt{20}}=-1.8\,.$$

(c) From Eq. (3.21),

$$\theta_{AC}=\cos^{-1}\frac{\mathbf{A}\cdot\mathbf{C}}{AC}=\cos^{-1}\frac{4+12}{\sqrt{14}\sqrt{20}}=\cos^{-1}\frac{16}{\sqrt{280}}=17.0^\circ.$$

(d) From Eq. (3.27),

$$\mathbf{A}\times\mathbf{C}=\hat{\mathbf{x}}(2(-4)-(-3)2)+\hat{\mathbf{y}}((-3)0-1(-4))+\hat{\mathbf{z}}(1(2)-2(0))=-\hat{\mathbf{x}}2+\hat{\mathbf{y}}4+\hat{\mathbf{z}}2.$$

(e) From Eq. (3.27) and Eq. (3.17),

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{A}\cdot(\hat{\mathbf{x}}16+\hat{\mathbf{y}}8+\hat{\mathbf{z}}4)=1(16)+2(8)+(-3)4=20.$$

Eq. (3.30) could also have been used in the solution. Also, Eq. (3.29) could be used in conjunction with the result of part (d).

(f) By repeated application of Eq. (3.27),

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{A}\times(\hat{\mathbf{x}}16+\hat{\mathbf{y}}8+\hat{\mathbf{z}}4)=\hat{\mathbf{x}}32-\hat{\mathbf{y}}52-\hat{\mathbf{z}}24.$$

Eq. (3.33) could also have been used.

(g) From Eq. (3.27),

$$\hat{\mathbf{x}} \times \mathbf{B} = -\hat{\mathbf{z}}4.$$

(h) From Eq. (3.27) and Eq. (3.17),

$$(\mathbf{A} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{z}} = (\hat{\mathbf{x}}3 + \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = 1.$$

Eq. (3.29) and Eq. (3.25) could also have been used in the solution.

Problem 3.6 Given vectors $\mathbf{A} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3$ and $\mathbf{B} = \hat{\mathbf{x}}3 - \hat{\mathbf{z}}2$, find a vector **C** whose magnitude is 9 and whose direction is perpendicular to both **A** and **B**.

Solution: The cross product of two vectors produces a new vector which is perpendicular to both of the original vectors. Two vectors exist which have a magnitude of 9 and are orthogonal to both **A** and **B**: one which is 9 units long in the direction of the unit vector parallel to $\mathbf{A} \times \mathbf{B}$, and one in the opposite direction.

$$\begin{aligned}\mathbf{C} = \pm 9 \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} &= \pm 9 \frac{(\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3) \times (\hat{\mathbf{x}}3 - \hat{\mathbf{z}}2)}{|(\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3) \times (\hat{\mathbf{x}}3 - \hat{\mathbf{z}}2)|} \\ &= \pm 9 \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}13 + \hat{\mathbf{z}}3}{\sqrt{2^2 + 13^2 + 3^2}} \approx \pm(\hat{\mathbf{x}}1.34 + \hat{\mathbf{y}}8.67 + \hat{\mathbf{z}}2.0).\end{aligned}$$

Problem 3.7 Given $\mathbf{A} = \hat{\mathbf{x}}(x + 2y) - \hat{\mathbf{y}}(y + 3z) + \hat{\mathbf{z}}(3x - y)$, determine a unit vector parallel to **A** at point $P(1, -1, 2)$.

Solution: The unit vector parallel to $\mathbf{A} = \hat{\mathbf{x}}(x + 2y) - \hat{\mathbf{y}}(y + 3z) + \hat{\mathbf{z}}(3x - y)$ at the point $P(1, -1, 2)$ is

$$\frac{\mathbf{A}(1,-1,2)}{|\mathbf{A}(1,-1,2)|} = \frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}}5 + \hat{\mathbf{z}}4}{\sqrt{(-1)^2 + (-5)^2 + 4^2}} = \frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}}5 + \hat{\mathbf{z}}4}{\sqrt{42}} \approx -\hat{\mathbf{x}}0.15 - \hat{\mathbf{y}}0.77 + \hat{\mathbf{z}}0.62.$$

Problem 3.8 By expansion in Cartesian coordinates, prove:
(a) the relation for the scalar triple product given by (3.29), and
(b) the relation for the vector triple product given by (3.33).

Solution:
(a) Proof of the scalar triple product given by Eq. (3.29): From Eq. (3.27),

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) + \hat{\mathbf{z}}(A_xB_y - A_yB_x),$$

$$\begin{aligned}\mathbf{B}\times\mathbf{C} &= \hat{\mathbf{x}}(B_yC_z - B_zC_y) + \hat{\mathbf{y}}(B_zC_x - B_xC_z) + \hat{\mathbf{z}}(B_xC_y - B_yC_x),\\ \mathbf{C}\times\mathbf{A} &= \hat{\mathbf{x}}(C_yA_z - C_zA_y) + \hat{\mathbf{y}}(C_zA_x - C_xA_z) + \hat{\mathbf{z}}(C_xA_y - C_yA_x).\end{aligned}$$

Employing Eq. (3.17), it is easily shown that

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x),\\ \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) &= B_x(C_yA_z - C_zA_y) + B_y(C_zA_x - C_xA_z) + B_z(C_xA_y - C_yA_x),\\ \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) &= C_x(A_yB_z - A_zB_y) + C_y(A_zB_x - A_xB_z) + C_z(A_xB_y - A_yB_x),\end{aligned}$$

which are all the same.

(b) Proof of the vector triple product given by Eq. (3.33): The evaluation of the left hand side employs the expression above for $\mathbf{B}\times\mathbf{C}$ with Eq. (3.27):

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\times(\hat{\mathbf{x}}(B_yC_z - B_zC_y) + \hat{\mathbf{y}}(B_zC_x - B_xC_z) + \hat{\mathbf{z}}(B_xC_y - B_yC_x))\\ &= \hat{\mathbf{x}}(A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z))\\ &\quad + \hat{\mathbf{y}}(A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x))\\ &\quad + \hat{\mathbf{z}}(A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)),\end{aligned}$$

while the right hand side, evaluated with the aid of Eq. (3.17), is

$$\begin{aligned}\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{B}(A_xC_x + A_yC_y + A_zC_z) - \mathbf{C}(A_xB_x + A_yB_y + A_zB_z)\\ &= \hat{\mathbf{x}}(B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z))\\ &\quad + \hat{\mathbf{y}}(B_y(A_xC_x + A_zC_z) - C_y(A_xB_x + A_zB_z))\\ &\quad + \hat{\mathbf{z}}(B_z(A_xC_x + A_yC_y) - C_z(A_xB_x + A_yB_y)).\end{aligned}$$

By rearranging the expressions for the components, the left hand side is equal to the right hand side.

Problem 3.9 Find an expression for the unit vector directed toward the origin from an arbitrary point on the line described by $x = 1$ and $z = 2$.

Solution: An arbitrary point on the given line is $(1, y, 2)$. The vector from this point to $(0, 0, 0)$ is:

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{x}}(0-1) + \hat{\mathbf{y}}(0-y) + \hat{\mathbf{z}}(0-2) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}y - 2\hat{\mathbf{z}},\\ |\mathbf{A}| &= \sqrt{1+y^2+4} = \sqrt{5+y^2},\\ \hat{\mathbf{a}} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}}y - \hat{\mathbf{z}}2}{\sqrt{5+y^2}}.\end{aligned}$$

Problem 3.10 Find an expression for the unit vector directed toward the point P located on the z-axis at a height h above the x–y plane from an arbitrary point $Q(x,y,-3)$ in the plane $z=-3$.

Solution: Point P is at $(0,0,h)$. Vector $\mathbf{A}$ from $Q(x,y,-3)$ to $P(0,0,h)$ is:

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{x}}(0-x)+\hat{\mathbf{y}}(0-y)+\hat{\mathbf{z}}(h+3) = -\hat{\mathbf{x}}x-\hat{\mathbf{y}}y+\hat{\mathbf{z}}(h+3),\\ |\mathbf{A}| &= [x^2+y^2+(h+3)^2]^{1/2},\\ \hat{\mathbf{a}} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{-\hat{\mathbf{x}}x-\hat{\mathbf{y}}y+\hat{\mathbf{z}}(h+3)}{[x^2+y^2+(h+3)^2]^{1/2}}\,.\end{aligned}$$

Problem 3.11 Find a unit vector parallel to either direction of the line described by

$$2x+z=4.$$

Solution: First, we find any two points on the given line. Since the line equation is not a function of y, the given line is in a plane parallel to the x–z plane. For convenience, we choose the x–z plane with $y=0$.

For $x=0$, $z=4$. Hence, point P is at $(0,0,4)$.

For $z=0$, $x=2$. Hence, point Q is at $(2,0,0)$.

Vector $\mathbf{A}$ from P to Q is:

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{x}}(2-0)+\hat{\mathbf{y}}(0-0)+\hat{\mathbf{z}}(0-4) = \hat{\mathbf{x}}2-\hat{\mathbf{z}}4,\\ \hat{\mathbf{a}} &= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}}2-\hat{\mathbf{z}}4}{\sqrt{20}}\,.\end{aligned}$$

Problem 3.12 Two lines in the x–y plane are described by the expressions:

$$\begin{array}{ll}\text{Line 1} & x+2y=-6,\\ \text{Line 2} & 3x+4y=8.\end{array}$$

Use vector algebra to find the smaller angle between the lines at their intersection point.

Solution: Intersection point is found by solving the two equations simultaneously:

$$\begin{aligned}-2x-4y &= 12,\\ 3x+4y &= 8.\end{aligned}$$

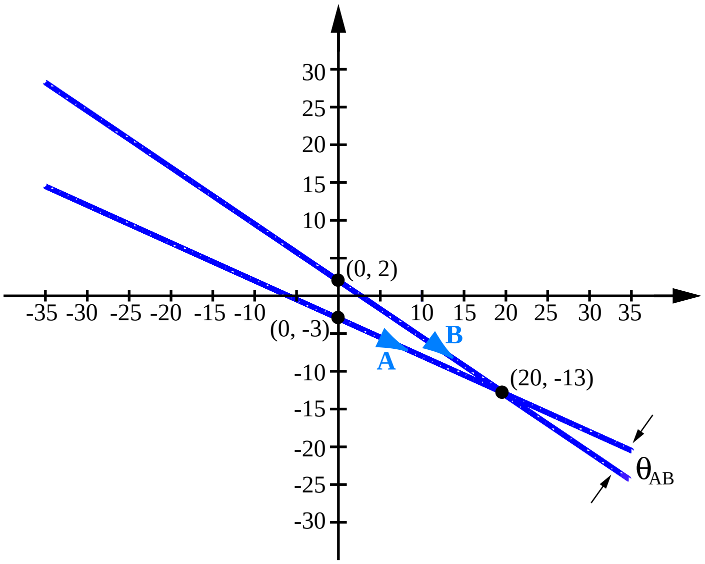


Figure P3.12: Lines 1 and 2.

The sum gives $x = 20$, which, when used in the first equation, gives $y = -13$.

Hence, intersection point is $(20, -13)$.

Another point on line 1 is $x = 0,\ y = -3$. Vector **A** from $(0, -3)$ to $(20, -13)$ is

$$\mathbf{A} = \hat{\mathbf{x}}(20) + \hat{\mathbf{y}}(-13+3) = \hat{\mathbf{x}}20 - \hat{\mathbf{y}}10,$$
$$|\mathbf{A}| = \sqrt{20^2 + 10^2} = \sqrt{500}.$$

A point on line 2 is $x = 0,\ y = 2$. Vector **B** from $(0, 2)$ to $(20, -13)$ is

$$\mathbf{B} = \hat{\mathbf{x}}(20) + \hat{\mathbf{y}}(-13-2) = \hat{\mathbf{x}}20 - \hat{\mathbf{y}}15,$$
$$|\mathbf{B}| = \sqrt{20^2 + 15^2} = \sqrt{625}.$$

Angle between **A** and **B** is

$$\theta_{AB} = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right) = \cos^{-1}\left(\frac{400+150}{\sqrt{500}\cdot\sqrt{625}}\right) = 10.3^\circ.$$

Problem 3.13 A given line is described by

$$x + 2y = 4.$$

Vector **A** starts at the origin and ends at point P on the line such that **A** is orthogonal to the line. Find an expression for **A**.

Solution: We first plot the given line. Next we find vector **B** which connects point $P_1(0,2)$ to $P_2(4,0)$, both of which are on the line:

$$\mathbf{B} = \hat{\mathbf{x}}(4-0) + \hat{\mathbf{y}}(0-2) = \hat{\mathbf{x}}4 - \hat{\mathbf{y}}2.$$

Vector **A** starts at the origin and ends on the line at P. If the x-coordinate of P is x,

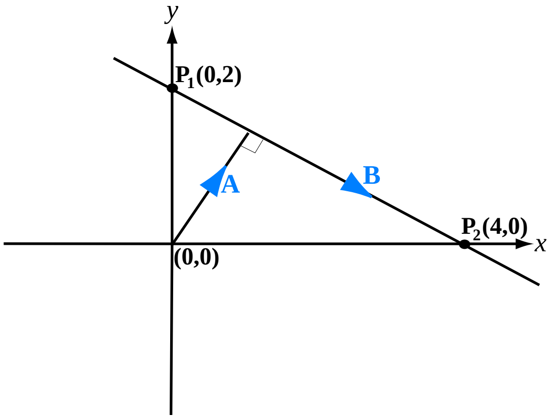


Figure P3.13: Given line and vector **A**.

then its y-coordinate has to be $(4-x)/2$ in order to be on the line. Hence P is at $(x,\ (4-x)/2)$. Vector **A** is

$$\mathbf{A} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}\left(\frac{4-x}{2}\right).$$

But **A** is perpendicular to the line. Hence,

$$\mathbf{A}\cdot\mathbf{B} = 0,$$

$$\left[\hat{\mathbf{x}}x + \hat{\mathbf{y}}\left(\frac{4-x}{2}\right)\right]\cdot(\hat{\mathbf{x}}4 - \hat{\mathbf{y}}2) = 0,$$

$$4x - (4-x) = 0, \quad \text{or}$$

$$x = \frac{4}{5} = 0.8.$$

Hence,

$$\mathbf{A} = \hat{\mathbf{x}}0.8 + \hat{\mathbf{y}}\left(\frac{4-0.8}{2}\right) = \hat{\mathbf{x}}0.8 + \hat{\mathbf{y}}1.6.$$

Problem 3.14 Show that, given two vectors **A** and **B**,

(a) the vector **C** defined as the vector component of **B** in the direction of **A** is given by

$$\mathbf{C} = \hat{\mathbf{a}}(\mathbf{B}\cdot\hat{\mathbf{a}}) = \frac{\mathbf{A}(\mathbf{B}\cdot\mathbf{A})}{|\mathbf{A}|^2},$$

where $\hat{\mathbf{a}}$ is the unit vector of **A**, and

(b) the vector **D** defined as the vector component of **B** perpendicular to **A** is given by

$$\mathbf{D} = \mathbf{B} - \frac{\mathbf{A}(\mathbf{B}\cdot\mathbf{A})}{|\mathbf{A}|^2}.$$

Solution:

(a) By definition, $\mathbf{B}\cdot\hat{\mathbf{a}}$ is the component of **B** along $\hat{\mathbf{a}}$. The vector component of $(\mathbf{B}\cdot\hat{\mathbf{a}})$ along **A** is

$$\mathbf{C} = \hat{\mathbf{a}}(\mathbf{B}\cdot\hat{\mathbf{a}}) = \frac{\mathbf{A}}{|\mathbf{A}|}\left(\mathbf{B}\cdot\frac{\mathbf{A}}{|\mathbf{A}|}\right) = \frac{\mathbf{A}(\mathbf{B}\cdot\mathbf{A})}{|\mathbf{A}|^2}.$$

(b) The figure shows vectors **A**, **B**, and **C**, where **C** is the projection of **B** along **A**. It is clear from the triangle that

$$\mathbf{B} = \mathbf{C} + \mathbf{D},$$

or

$$\mathbf{D} = \mathbf{B} - \mathbf{C} = \mathbf{B} - \frac{\mathbf{A}(\mathbf{B}\cdot\mathbf{A})}{|\mathbf{A}|^2}.$$

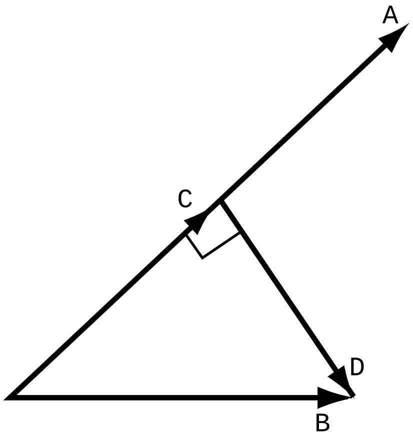


Figure P3.14: Relationships between vectors **A**, **B**, **C**, and **D**.

Problem 3.15 A certain plane is described by

$$2x+3y+4z=16.$$

Find the unit vector normal to the surface in the direction away from the origin.

Solution: Procedure:

1. Use the equation for the given plane to find three points, P_1, P_2 and P_3 on the plane.
2. Find vector **A** from P_1 to P_2 and vector **B** from P_1 to P_3.
3. Cross product of **A** and **B** gives a vector **C** orthogonal to **A** and **B**, and hence to the plane.
4. Check direction of $\hat{\mathbf{c}}$.

Steps:

1. Choose the following three points:

$$\begin{aligned} &P_1 \text{ at } (0,0,4),\\ &P_2 \text{ at } (8,0,0),\\ &P_3 \text{ at } (0,\tfrac{16}{3},0). \end{aligned}$$

2. Vector **A** from P_1 to P_2

$$\mathbf{A}=\hat{\mathbf{x}}\,(8-0)+\hat{\mathbf{y}}\,(0-0)+\hat{\mathbf{z}}\,(0-4)=\hat{\mathbf{x}}\,8-\hat{\mathbf{z}}\,4$$

Vector **B** from P_1 to P_3

$$\mathbf{B}=\hat{\mathbf{x}}\,(0-0)+\hat{\mathbf{y}}\left(\frac{16}{3}-0\right)+\hat{\mathbf{z}}\,(0-4)=\hat{\mathbf{y}}\,\frac{16}{3}-\hat{\mathbf{z}}\,4$$

3.

$$\begin{aligned}\mathbf{C}&=\mathbf{A}\times\mathbf{B}\\ &=\hat{\mathbf{x}}\,(A_yB_z-A_zB_y)+\hat{\mathbf{y}}\,(A_zB_x-A_xB_z)+\hat{\mathbf{z}}\,(A_xB_y-A_yB_x)\\ &=\hat{\mathbf{x}}\left(0\cdot(-4)-(-4)\cdot\frac{16}{3}\right)+\hat{\mathbf{y}}\,((-4)\cdot 0-8\cdot(-4))+\hat{\mathbf{z}}\left(8\cdot\frac{16}{3}-0\cdot 0\right)\\ &=\hat{\mathbf{x}}\,\frac{64}{3}+\hat{\mathbf{y}}\,32+\hat{\mathbf{z}}\,\frac{128}{3}\end{aligned}$$

Verify that **C** is orthogonal to **A** and **B**

$$\mathbf{A}\cdot\mathbf{C} = \left(8\cdot\frac{64}{3}\right) + (32\cdot 0) + \left(\frac{128}{3}\cdot(-4)\right) = \frac{512}{3} - \frac{512}{3} = 0$$
$$\mathbf{B}\cdot\mathbf{C} = \left(0\cdot\frac{64}{3}\right) + \left(32\cdot\frac{16}{3}\right) + \left(\frac{128}{3}\cdot(-4)\right) = \frac{512}{3} - \frac{512}{3} = 0$$

4. $\mathbf{C} = \hat{\mathbf{x}}\,\frac{64}{3} + \hat{\mathbf{y}}\,32 + \hat{\mathbf{z}}\,\frac{128}{3}$

$$\hat{\mathbf{c}} = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{\hat{\mathbf{x}}\,\frac{64}{3} + \hat{\mathbf{y}}\,32 + \hat{\mathbf{z}}\,\frac{128}{3}}{\sqrt{\left(\frac{64}{3}\right)^2 + 32^2 + \left(\frac{128}{3}\right)^2}} = \hat{\mathbf{x}}\,0.37 + \hat{\mathbf{y}}\,0.56 + \hat{\mathbf{z}}\,0.74.$$

$\hat{\mathbf{c}}$ points away from the origin as desired.

Problem 3.16 Given $\mathbf{B} = \hat{\mathbf{x}}(z-3y) + \hat{\mathbf{y}}(2x-3z) - \hat{\mathbf{z}}(x+y)$, find a unit vector parallel to **B** at point $P(1,0,-1)$.

Solution: At $P(1,0,-1)$,

$$\mathbf{B} = \hat{\mathbf{x}}(-1) + \hat{\mathbf{y}}(2+3) - \hat{\mathbf{z}}(1) = -\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}},$$
$$\hat{\mathbf{b}} = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}}{\sqrt{1+25+1}} = \frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}}{\sqrt{27}}.$$

Problem 3.17 When sketching or demonstrating the spatial variation of a vector field, we often use arrows, as in Fig. 3-25 (P3.17), wherein the length of the arrow is made to be proportional to the strength of the field and the direction of the arrow is the same as that of the field's. The sketch shown in Fig. P3.17, which represents the vector field $\mathbf{E} = \hat{\mathbf{r}}r$, consists of arrows pointing radially away from the origin and their lengths increase linearly in proportion to their distance away from the origin. Using this arrow representation, sketch each of the following vector fields:

(a) $\mathbf{E}_1 = -\hat{\mathbf{x}}y$,

(b) $\mathbf{E}_2 = \hat{\mathbf{y}}x$,

(c) $\mathbf{E}_3 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$,

(d) $\mathbf{E}_4 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}2y$,

(e) $\mathbf{E}_5 = \hat{\boldsymbol{\phi}}r$,

(f) $\mathbf{E}_6 = \hat{\mathbf{r}}\sin\phi$.

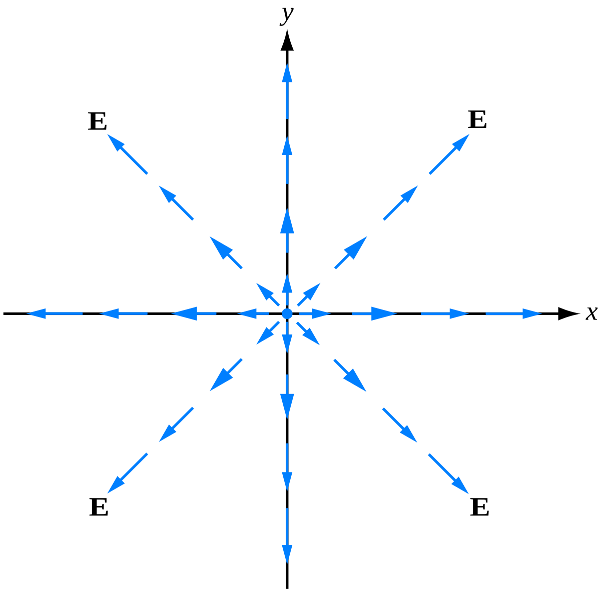
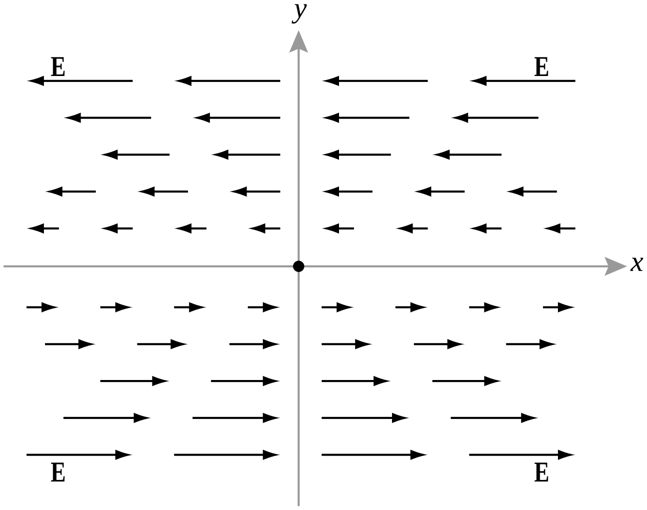


Figure P3.17: Arrow representation for vector field $\mathbf{E} = \hat{\mathbf{r}}r$ (Problem 3.17).

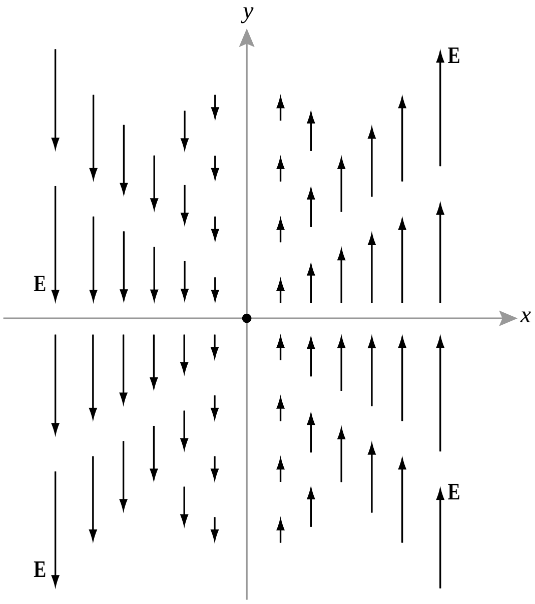
Solution:

(a)



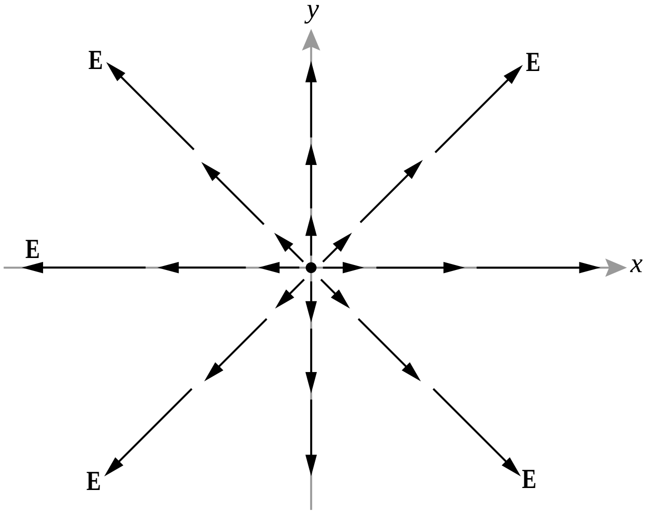
P2.13a: $\mathbf{E}_1 = -\hat{\mathbf{x}}\, y$

(b)



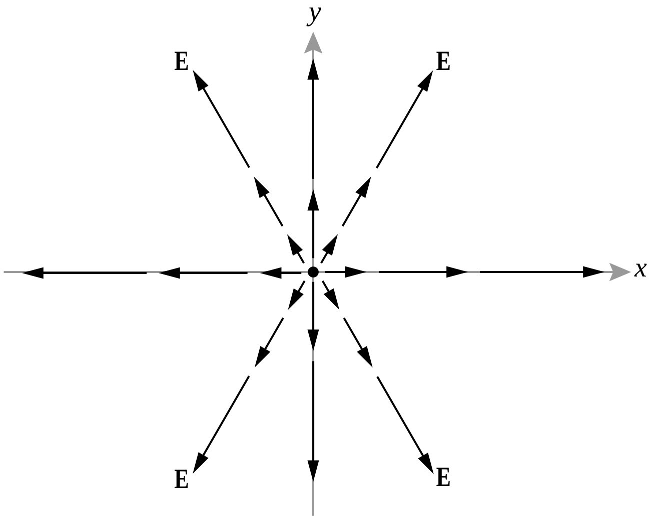
P3.17b: $\mathbf{E}_2 = \hat{\mathbf{y}}x$

(c)



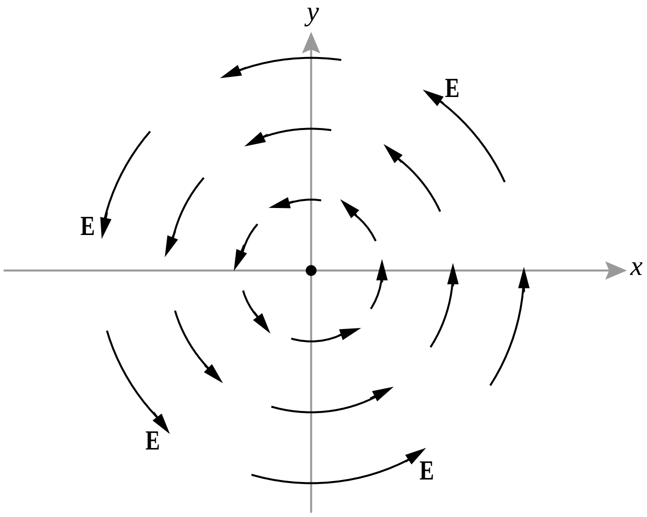
P2.13c: $\mathbf{E}_3 = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$

(d)



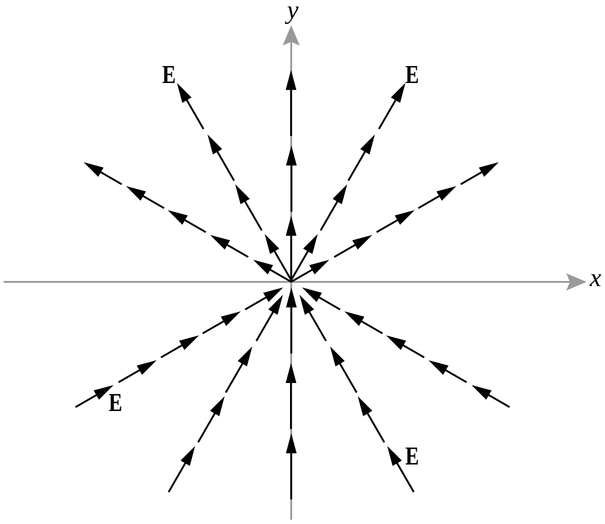
P2.13d: $\mathbf{E}_4 = \hat{\mathbf{x}}\,x + \hat{\mathbf{y}}\,2y$

(e)



P2.13e: $\mathbf{E}_5 = \hat{\boldsymbol{\phi}}\,r$

(f)



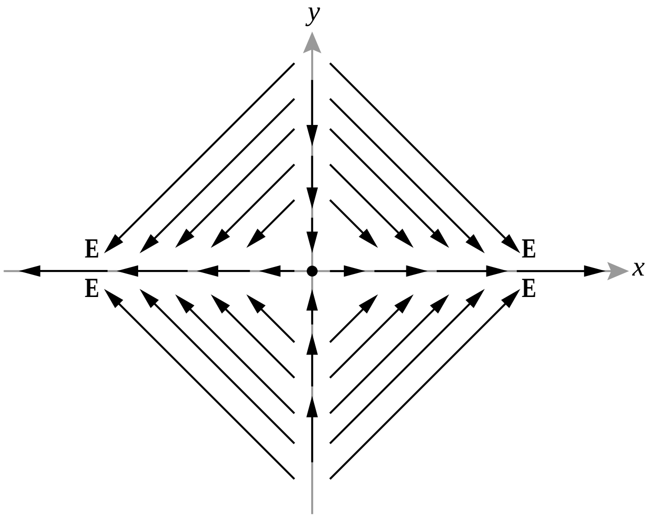
P2.13f: $\mathbf{E}_6 = \hat{\mathbf{r}} \sin\phi$

Problem 3.18 Use arrows to sketch each of the following vector fields:

- **(a)** $\mathbf{E}_1 = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$,
- **(b)** $\mathbf{E}_2 = -\hat{\boldsymbol{\phi}}$,
- **(c)** $\mathbf{E}_3 = \hat{\mathbf{y}}\frac{1}{x}$,
- **(d)** $\mathbf{E}_4 = \hat{\mathbf{r}}\cos\phi$.

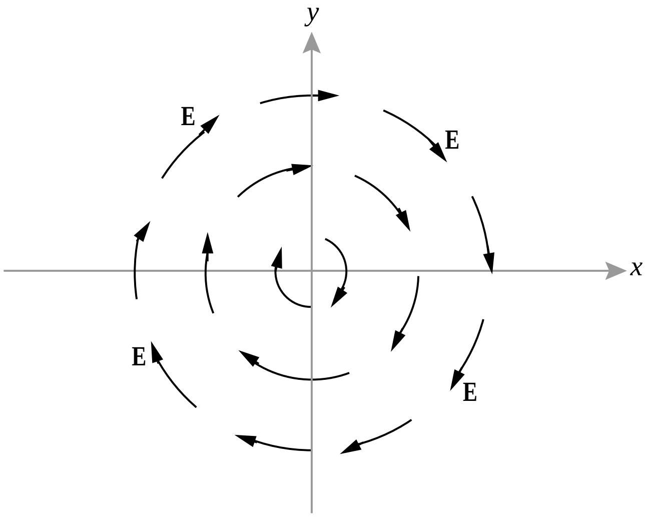
Solution:

(a)



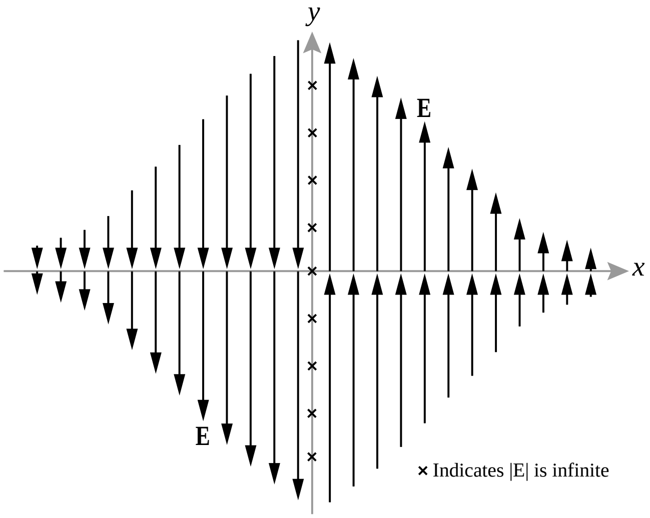
P2.14a: $\mathbf{E}_1 = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$

(b)



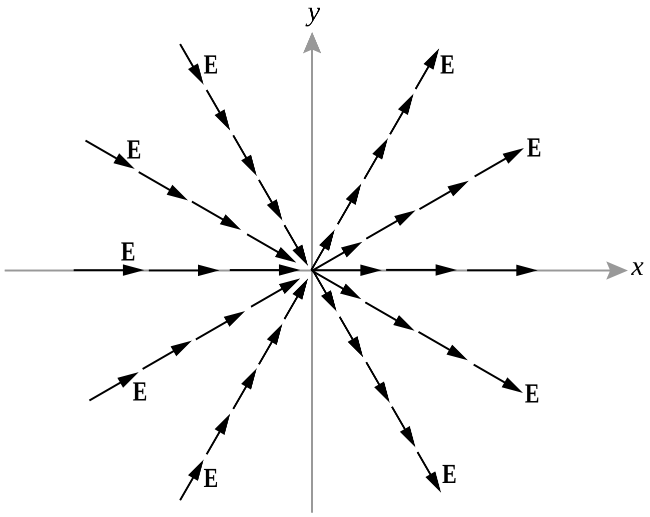
P2.14b: $\mathbf{E}_2 = -\hat{\boldsymbol{\phi}}$

(c)



P2.14c: $\mathbf{E}_3 = \hat{\mathbf{y}}(1/x)$

(d)



P2.14d: $\mathbf{E}_4 = \hat{\mathbf{r}}\cos\phi$

Sections 3-2 and 3-3: Coordinate Systems

Problem 3.19 Convert the coordinates of the following points from Cartesian to cylindrical and spherical coordinates:

(a) $P_1(1,2,0)$,
(b) $P_2(0,0,2)$,
(c) $P_3(1,1,3)$,
(d) $P_4(-2,2,-2)$.

Solution: Use the "coordinate variables" column in Table 3-2.

(a) In the cylindrical coordinate system,

$$P_1 = (\sqrt{1^2+2^2}, \tan^{-1}(2/1), 0) = (\sqrt{5}, 1.107 \text{ rad}, 0) \approx (2.24, 63.4^\circ, 0).$$

In the spherical coordinate system,

$$\begin{aligned} P_1 &= (\sqrt{1^2+2^2+0^2}, \tan^{-1}(\sqrt{1^2+2^2}/0), \tan^{-1}(2/1)) \\ &= (\sqrt{5}, \pi/2 \text{ rad}, 1.107 \text{ rad}) \approx (2.24, 90.0^\circ, 63.4^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant I.

(b) In the cylindrical coordinate system,

$$P_2 = (\sqrt{0^2+0^2}, \tan^{-1}(0/0), 2) = (0, 0 \text{ rad}, 2) = (0, 0^\circ, 2).$$

In the spherical coordinate system,

$$\begin{aligned} P_2 &= (\sqrt{0^2+0^2+2^2}, \tan^{-1}(\sqrt{0^2+0^2}/2), \tan^{-1}(0/0)) \\ &= (2, 0 \text{ rad}, 0 \text{ rad}) = (2, 0^\circ, 0^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is arbitrary and may take any value.

(c) In the cylindrical coordinate system,

$$P_3 = (\sqrt{1^2+1^2}, \tan^{-1}(1/1), 3) = (\sqrt{2}, \pi/4 \text{ rad}, 3) \approx (1.41, 45.0^\circ, 3).$$

In the spherical coordinate system,

$$\begin{aligned} P_3 &= (\sqrt{1^2+1^2+3^2}, \tan^{-1}(\sqrt{1^2+1^2}/3), \tan^{-1}(1/1)) \\ &= (\sqrt{11}, 0.44 \text{ rad}, \pi/4 \text{ rad}) \approx (3.32, 25.2^\circ, 45.0^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant I.

(d) In the cylindrical coordinate system,

$$\begin{aligned} P_4 &= (\sqrt{(-2)^2+2^2}, \tan^{-1}(2/-2), -2) \\ &= (2\sqrt{2}, 3\pi/4 \text{ rad}, -2) \approx (2.83, 135.0^\circ, -2). \end{aligned}$$

In the spherical coordinate system,

$$\begin{aligned} P_4 &= (\sqrt{(-2)^2+2^2+(-2)^2}, \tan^{-1}(\sqrt{(-2)^2+2^2}/-2), \tan^{-1}(2/-2)) \\ &= (2\sqrt{3}, 2.187 \text{ rad}, 3\pi/4 \text{ rad}) \approx (3.46, 125.3^\circ, 135.0^\circ). \end{aligned}$$

Note that in both the cylindrical and spherical coordinates, ϕ is in Quadrant II.

Problem 3.20 Convert the coordinates of the following points from cylindrical to Cartesian coordinates:

(a) $P_1(2, \pi/4, -2)$,

(b) $P_2(3, 0, -2)$,

(c) $P_3(4, \pi, 3)$.

Solution:

(a)

$$P_1(x,y,z) = P_1(r\cos\phi, r\sin\phi, z) = P_1\left(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}, -2\right) = P_1(1.41, 1.41, -2).$$

(b) $P_2(x,y,z) = P_2(3\cos 0, 3\sin 0, -2) = P_2(3, 0, -2)$.

(c) $P_3(x,y,z) = P_3(4\cos\pi, 4\sin\pi, 3) = P_3(-4, 0, 3)$.

Problem 3.21 Convert the coordinates of the following points from spherical to cylindrical coordinates:

(a) $P_1(5, 0, 0)$,

(b) $P_2(5, 0, \pi)$,

(c) $P_3(3, \pi/2, 0)$.

Solution:

(a)

$$\begin{aligned} P_1(r,\phi,z) = P_1(R\sin\theta, \phi, R\cos\theta) &= P_1(5\sin 0, 0, 5\cos 0) \\ &= P_1(0, 0, 5). \end{aligned}$$

(b) $P_2(r,\phi,z) = P_2(5\sin 0, \pi, 5\cos 0) = P_2(0, \pi, 5)$.

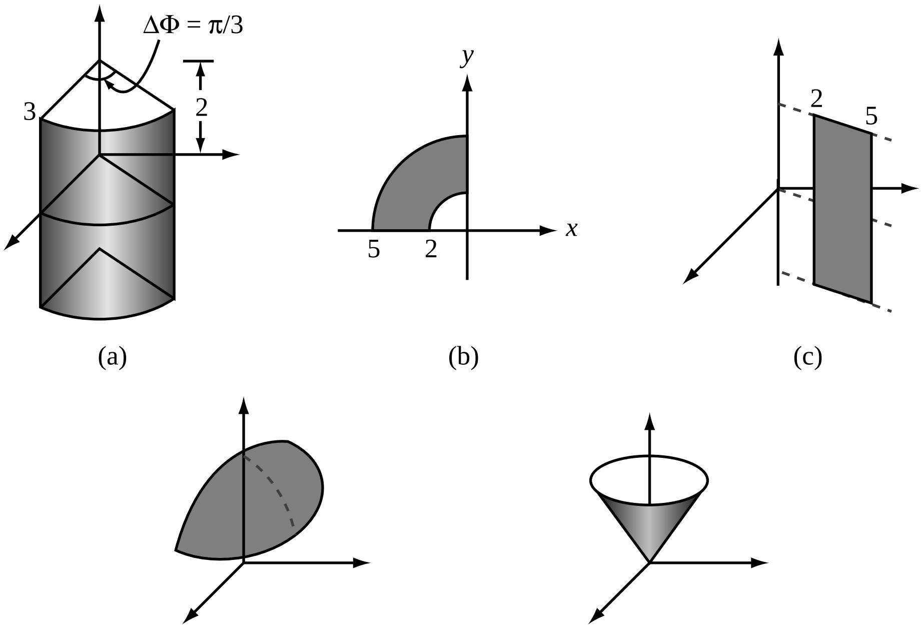
(c) $P_3(r,\phi,z) = P_3(3\sin\frac{\pi}{2}, 0, 3\cos\frac{\pi}{2}) = P_3(3,0,0)$.

Problem 3.22 Use the appropriate expression for the differential surface area $d\mathbf{s}$ to determine the area of each of the following surfaces:

(a) $r = 3;\ 0 \le \phi \le \pi/3;\ -2 \le z \le 2$,
(b) $2 \le r \le 5;\ \pi/2 \le \phi \le \pi;\ z = 0$,
(c) $2 \le r \le 5;\ \phi = \pi/4;\ -2 \le z \le 2$,
(d) $R = 2;\ 0 \le \theta \le \pi/3;\ 0 \le \phi \le \pi$,
(e) $0 \le R \le 5;\ \theta = \pi/3;\ 0 \le \phi \le 2\pi$.

Also sketch the outlines of each of the surfaces.

Solution:



(a) (b) (c)

(d) (e)

Figure P3.22: Surfaces described by Problem 3.22.

(a) Using Eq. (3.43a),

$$A = \int_{z=-2}^{2}\int_{\phi=0}^{\pi/3} (r)\big|_{r=3}\, d\phi\, dz = \left((3\phi z)\big|_{\phi=0}^{\pi/3} \right)\Big|_{z=-2}^{2} = 4\pi.$$

(b) Using Eq. (3.43c),

$$A=\int_{r=2}^{5}\int_{\phi=\pi/2}^{\pi}(r)|_{z=0}\,d\phi\,dr=\left(\left(\tfrac{1}{2}r^2\phi\right)\Big|_{r=2}^{5}\right)\Big|_{\phi=\pi/2}^{\pi}=\frac{21\pi}{4}.$$

(c) Using Eq. (3.43b),

$$A=\int_{z=-2}^{2}\int_{r=2}^{5}(1)|_{\phi=\pi/4}\,dr\,dz=\left((rz)|_{z=-2}^{2}\right)\Big|_{r=2}^{5}=12.$$

(d) Using Eq. (3.50b),

$$A=\int_{\theta=0}^{\pi/3}\int_{\phi=0}^{\pi}\left(R^2\sin\theta\right)\Big|_{R=2}\,d\phi\,d\theta=\left((-4\phi\cos\theta)|_{\theta=0}^{\pi/3}\right)\Big|_{\phi=0}^{\pi}=2\pi.$$

(e) Using Eq. (3.50c),

$$A=\int_{R=0}^{5}\int_{\phi=0}^{2\pi}(R\sin\theta)|_{\theta=\pi/3}\,d\phi\,dR=\left(\left(\tfrac{1}{2}R^2\phi\sin\frac{\pi}{3}\right)\Big|_{\phi=0}^{2\pi}\right)\Big|_{R=0}^{5}=\frac{25\sqrt{3}\pi}{2}.$$

Problem 3.23 Find the volumes described by

(a) $2\le r\le 5;\ \pi/2\le\phi\le\pi;\ 0\le z\le 2,$

(b) $0\le R\le 5;\ 0\le\theta\le\pi/3;\ 0\le\phi\le 2\pi.$

Also sketch the outline of each volume.

Solution:

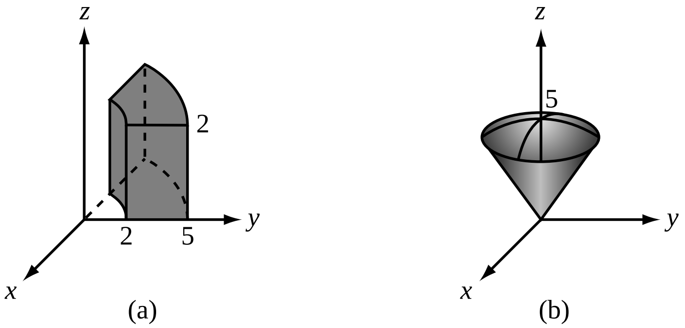


Figure P3.23: Volumes described by Problem 3.23 .

(a) From Eq. (3.44),

$$V=\int_{z=0}^{2}\int_{\phi=\pi/2}^{\pi}\int_{r=2}^{5}r\,dr\,d\phi\,dz=\left(\left(\left(\tfrac{1}{2}r^2\phi z\right)\Big|_{r=2}^{5}\right)\Big|_{\phi=\pi/2}^{\pi}\right)\Big|_{z=0}^{2}=\frac{21\pi}{2}.$$

(b) From Eq. (3.50e),

$$V = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/3} \int_{R=0}^{5} R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= \left(\left(\left(-\cos\theta \frac{R^3}{3} \phi \right) \Bigg|_{R=0}^{5} \right) \Bigg|_{\theta=0}^{\pi/3} \right) \Bigg|_{\phi=0}^{2\pi} = \frac{125\pi}{3}.$$

Problem 3.24 A section of a sphere is described by $0 \le R \le 2$, $0 \le \theta \le 90°$, and $30° \le \phi \le 90°$. Find:

(a) the surface area of the spherical section,

(b) the enclosed volume.

Also sketch the outline of the section.

Solution:

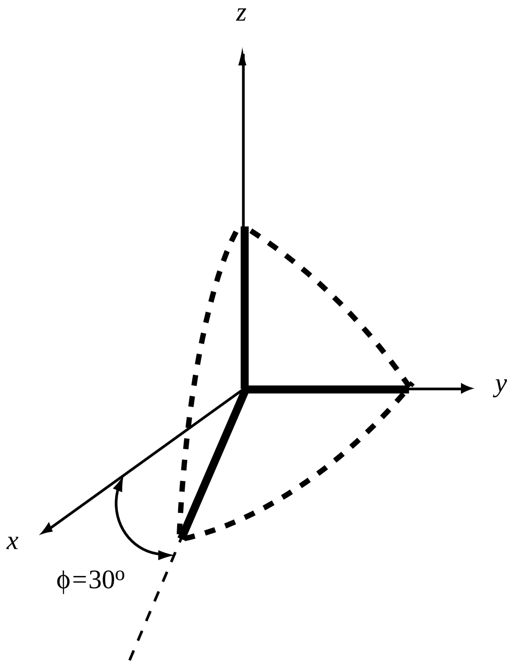


Figure P3.24: Outline of section.

$$\begin{aligned}
S &= \int_{\phi=\pi/6}^{\pi/2} \int_{\theta=0}^{\pi/2} R^2 \sin\theta \, d\theta \, d\phi |_{R=2} \\
&= 4\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \left[-\cos\theta|_0^{\pi/2}\right] = 4 \times \frac{\pi}{3} = \frac{4\pi}{3} \quad (\text{m}^2), \\
V &= \int_{R=0}^{2} \int_{\phi=\pi/6}^{\pi/2} \int_{\theta=0}^{\pi/2} R^2 \sin\theta \, dR \, d\theta \, d\phi \\
&= \frac{R^3}{3}\bigg|_0^2 \left(\frac{\pi}{2} - \frac{\pi}{6}\right) [-\cos\theta|_0^{\pi/2}] = \frac{8}{3}\frac{\pi}{3} = \frac{8\pi}{9} \quad (\text{m}^3).
\end{aligned}$$

Problem 3.25 A vector field is given in cylindrical coordinates by

$$\mathbf{E} = \hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}} r\sin\phi + \hat{\mathbf{z}} z^2.$$

Point $P(2,\pi,3)$ is located on the surface of the cylinder described by $r = 2$. At point P, find:

(a) the vector component of **E** perpendicular to the cylinder,

(b) the vector component of **E** tangential to the cylinder.

Solution:

(a) $\mathbf{E}_\text{n} = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{E}) = \hat{\mathbf{r}}[\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}} r\sin\phi + \hat{\mathbf{z}} z^2)] = \hat{\mathbf{r}} r\cos\phi$.
At $P(2,\pi,3)$, $\mathbf{E}_\text{n} = \hat{\mathbf{r}} 2\cos\pi = -\hat{\mathbf{r}} 2$.
(b) $\mathbf{E}_\text{t} = \mathbf{E} - \mathbf{E}_\text{n} = \hat{\boldsymbol{\phi}} r\sin\phi + \hat{\mathbf{z}} z^2$.
At $P(2,\pi,3)$, $\mathbf{E}_\text{t} = \hat{\boldsymbol{\phi}} 2\sin\pi + \hat{\mathbf{z}} 3^2 = \hat{\mathbf{z}} 9$.

Problem 3.26 At a given point in space, vectors **A** and **B** are given in spherical coordinates by

$$\mathbf{A} = \hat{\mathbf{R}} 4 + \hat{\boldsymbol{\theta}} 2 - \hat{\boldsymbol{\phi}},$$
$$\mathbf{B} = -\hat{\mathbf{R}} 2 + \hat{\boldsymbol{\phi}} 3.$$

Find:

(a) the scalar component, or projection, of **B** in the direction of **A**,

(b) the vector component of **B** in the direction of **A**,

(c) the vector component of **B** perpendicular to **A**.

Solution:

(a) Scalar component of **B** in direction of **A**:

$$C = \mathbf{B}\cdot\hat{\mathbf{a}} = \mathbf{B}\cdot\frac{\mathbf{A}}{|\mathbf{A}|} = (-\hat{\mathbf{R}}2+\hat{\boldsymbol{\phi}}3)\cdot\frac{(\hat{\mathbf{R}}4+\hat{\boldsymbol{\theta}}2-\hat{\boldsymbol{\phi}})}{\sqrt{16+4+1}}$$
$$= \frac{-8-3}{\sqrt{21}} = -\frac{11}{\sqrt{21}} = -2.4.$$

(b) Vector component of **B** in direction of **A**:

$$\mathbf{C} = \hat{\mathbf{a}}C = \mathbf{A}\frac{C}{|\mathbf{A}|} = (\hat{\mathbf{R}}4+\hat{\boldsymbol{\theta}}2-\hat{\boldsymbol{\phi}})\,\frac{(-2.4)}{\sqrt{21}}$$
$$= -(\hat{\mathbf{R}}2.09+\hat{\boldsymbol{\theta}}1.05-\hat{\boldsymbol{\phi}}0.52).$$

(c) Vector component of **B** perpendicular to **A**:

$$\mathbf{D} = \mathbf{B}-\mathbf{C} = (-\hat{\mathbf{R}}2+\hat{\boldsymbol{\phi}}3)+(\hat{\mathbf{R}}2.09+\hat{\boldsymbol{\theta}}1.05-\hat{\boldsymbol{\phi}}0.52)$$
$$= \hat{\mathbf{R}}0.09+\hat{\boldsymbol{\theta}}1.05+\hat{\boldsymbol{\phi}}2.48.$$

Problem 3.27 Given vectors

$$\mathbf{A} = \hat{\mathbf{r}}(\cos\phi+3z)-\hat{\boldsymbol{\phi}}(2r+4\sin\phi)+\hat{\mathbf{z}}(r-2z),$$
$$\mathbf{B} = -\hat{\mathbf{r}}\sin\phi+\hat{\mathbf{z}}\cos\phi,$$

find

(a) θ_{AB} at $(2,\pi/2,0)$,

(b) a unit vector perpendicular to both **A** and **B** at $(2,\pi/3,1)$.

Solution: It doesn't matter whether the vectors are evaluated before vector products are calculated, or if the vector products are directly calculated and the general results are evaluated at the specific point in question.

(a) At $(2,\pi/2,0)$, $\mathbf{A} = -\hat{\boldsymbol{\phi}}8+\hat{\mathbf{z}}2$ and $\mathbf{B} = -\hat{\mathbf{r}}$. From Eq. (3.21),

$$\theta_{AB} = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) = \cos^{-1}\left(\frac{0}{AB}\right) = 90^\circ.$$

(b) At $(2,\pi/3,1)$, $\mathbf{A} = \hat{\mathbf{r}}\frac{7}{2}-\hat{\boldsymbol{\phi}}4(1+\frac{1}{2}\sqrt{3})$ and $\mathbf{B} = -\hat{\mathbf{r}}\frac{1}{2}\sqrt{3}+\hat{\mathbf{z}}\frac{1}{2}$. Since $\mathbf{A}\times\mathbf{B}$ is perpendicular to both **A** and **B**, a unit vector perpendicular to both **A** and **B** is given by

$$\pm\frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = \pm\frac{\hat{\mathbf{r}}(-4(1+\frac{1}{2}\sqrt{3}))(\frac{1}{2})-\hat{\boldsymbol{\phi}}(\frac{7}{2})(\frac{1}{2})-\hat{\mathbf{z}}(4(1+\frac{1}{2}\sqrt{3}))(\frac{1}{2}\sqrt{3})}{\sqrt{(2(1+\frac{1}{2}\sqrt{3}))^2+(\frac{7}{4})^2+(3+2\sqrt{3})^2}}$$
$$\approx \mp(\hat{\mathbf{r}}0.487+\hat{\boldsymbol{\phi}}0.228+\hat{\mathbf{z}}0.843).$$

Problem 3.28 Find the distance between the following pairs of points:

(a) $P_1(1,2,3)$ and $P_2(-2,-3,-2)$ in Cartesian coordinates,

(b) $P_3(1,\pi/4,3)$ and $P_4(3,\pi/4,4)$ in cylindrical coordinates,

(c) $P_5(4,\pi/2,0)$ and $P_6(3,\pi,0)$ in spherical coordinates.

Solution:

(a)

$$d = [(-2-1)^2 + (-3-2)^2 + (-2-3)^2]^{1/2} = [9+25+25]^{1/2} = \sqrt{59} = 7.68.$$

(b)

$$\begin{aligned} d &= [r_2^2 + r_1^2 - 2r_1r_2\cos(\phi_2-\phi_1) + (z_2-z_1)^2]^{1/2} \\ &= \left[9+1-2\times 3\times 1\times \cos\left(\frac{\pi}{4}-\frac{\pi}{4}\right) + (4-3)^2\right]^{1/2} \\ &= (10-6+1)^{1/2} = 5^{1/2} = 2.24. \end{aligned}$$

(c)

$$\begin{aligned} d &= \{R_2^2 + R_1^2 - 2R_1R_2[\cos\theta_2\cos\theta_1 + \sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1)]\}^{1/2} \\ &= \left\{9+16-2\times 3\times 4\left[\cos\pi\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\sin\pi\cos(0-0)\right]\right\}^{1/2} \\ &= \{9+16-0\}^{1/2} = \sqrt{25} = 5. \end{aligned}$$

Problem 3.29 Determine the distance between the following pairs of points:

(a) $P_1(1,1,2)$ and $P_2(0,2,3)$,

(b) $P_3(2,\pi/3,1)$ and $P_4(4,\pi/2,3)$,

(c) $P_5(3,\pi,\pi/2)$ and $P_6(4,\pi/2,\pi)$.

Solution:

(a) From Eq. (3.66),

$$d = \sqrt{(0-1)^2 + (2-1)^2 + (3-2)^2} = \sqrt{3}.$$

(b) From Eq. (3.67),

$$d = \sqrt{2^2 + 4^2 - 2(2)(4)\cos\left(\frac{\pi}{2}-\frac{\pi}{3}\right) + (3-1)^2} = \sqrt{24-8\sqrt{3}} \approx 3.18.$$

(c) From Eq. (3.68),

$$d = \sqrt{3^2+4^2-2(3)(4)\left(\cos\frac{\pi}{2}\cos\pi+\sin\pi\sin\frac{\pi}{2}\cos\left(\pi-\frac{\pi}{2}\right)\right)} = 5.$$

Problem 3.30 Transform the following vectors into cylindrical coordinates and then evaluate them at the indicated points:

(a) $\mathbf{A} = \hat{\mathbf{x}}(x+y)$ at $P_1(1,2,3)$,
(b) $\mathbf{B} = \hat{\mathbf{x}}(y-x)+\hat{\mathbf{y}}(x-y)$ at $P_2(1,0,2)$,
(c) $\mathbf{C} = \hat{\mathbf{x}}y^2/(x^2+y^2)-\hat{\mathbf{y}}x^2/(x^2+y^2)+\hat{\mathbf{z}}4$ at $P_3(1,-1,2)$,
(d) $\mathbf{D} = \hat{\mathbf{R}}\sin\theta+\hat{\boldsymbol{\theta}}\cos\theta+\hat{\boldsymbol{\phi}}\cos^2\phi$ at $P_4(2,\pi/2,\pi/4)$,
(e) $\mathbf{E} = \hat{\mathbf{R}}\cos\phi+\hat{\boldsymbol{\theta}}\sin\phi+\hat{\boldsymbol{\phi}}\sin^2\theta$ at $P_5(3,\pi/2,\pi)$.

Solution: From Table 3-2:

(a)

$$\begin{aligned}
\mathbf{A} &= (\hat{\mathbf{r}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi)(r\cos\phi+r\sin\phi)\\
&= \hat{\mathbf{r}}r\cos\phi(\cos\phi+\sin\phi)-\hat{\boldsymbol{\phi}}r\sin\phi(\cos\phi+\sin\phi),\\
P_1 &= (\sqrt{1^2+2^2},\tan^{-1}(2/1),3) = (\sqrt{5},63.4^\circ,3),\\
\mathbf{A}(P_1) &= (\hat{\mathbf{r}}0.447-\hat{\boldsymbol{\phi}}0.894)\sqrt{5}(.447+.894) = \hat{\mathbf{r}}1.34-\hat{\boldsymbol{\phi}}2.68.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{B} &= (\hat{\mathbf{r}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi)(r\sin\phi-r\cos\phi)+(\hat{\boldsymbol{\phi}}\cos\phi+\hat{\mathbf{r}}\sin\phi)(r\cos\phi-r\sin\phi)\\
&= \hat{\mathbf{r}}r(2\sin\phi\cos\phi-1)+\hat{\boldsymbol{\phi}}r(\cos^2\phi-\sin^2\phi) = \hat{\mathbf{r}}r(\sin 2\phi-1)+\hat{\boldsymbol{\phi}}r\cos 2\phi,\\
P_2 &= (\sqrt{1^2+0^2},\tan^{-1}(0/1),2) = (1,0^\circ,2),\\
\mathbf{B}(P_2) &= -\hat{\mathbf{r}}+\hat{\boldsymbol{\phi}}.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{C} &= (\hat{\mathbf{r}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi)\frac{r^2\sin^2\phi}{r^2}-(\hat{\boldsymbol{\phi}}\cos\phi+\hat{\mathbf{r}}\sin\phi)\frac{r^2\cos^2\phi}{r^2}+\hat{\mathbf{z}}4\\
&= \hat{\mathbf{r}}\sin\phi\cos\phi(\sin\phi-\cos\phi)-\hat{\boldsymbol{\phi}}(\sin^3\phi+\cos^3\phi)+\hat{\mathbf{z}}4,\\
P_3 &= (\sqrt{1^2+(-1)^2},\tan^{-1}(-1/1),2) = (\sqrt{2},-45^\circ,2),\\
\mathbf{C}(P_3) &= \hat{\mathbf{r}}0.707+\hat{\mathbf{z}}4.
\end{aligned}$$

(d)

$$\mathbf{D} = (\hat{\mathbf{r}}\sin\theta+\hat{\mathbf{z}}\cos\theta)\sin\theta+(\hat{\mathbf{r}}\cos\theta-\hat{\mathbf{z}}\sin\theta)\cos\theta+\hat{\boldsymbol{\phi}}\cos^2\phi = \hat{\mathbf{r}}+\hat{\boldsymbol{\phi}}\cos^2\phi,$$

$$\begin{aligned} P_4 &= (2\sin(\pi/2), \pi/4, 2\cos(\pi/2)) = (2, 45^\circ, 0), \\ \mathbf{D}(P_4) &= \hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}\tfrac{1}{2}\,. \end{aligned}$$

(e)

$$\begin{aligned} \mathbf{E} &= (\hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\cos\phi + (\hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta)\sin\phi + \hat{\boldsymbol{\phi}}\sin^2\theta, \\ P_5 &= \left(3, \frac{\pi}{2}, \pi\right), \\ \mathbf{E}(P_5) &= \left(\hat{\mathbf{r}}\sin\frac{\pi}{2} + \hat{\mathbf{z}}\cos\frac{\pi}{2}\right)\cos\pi + \left(\hat{\mathbf{r}}\cos\frac{\pi}{2} - \hat{\mathbf{z}}\sin\frac{\pi}{2}\right)\sin\pi + \hat{\boldsymbol{\phi}}\sin^2\frac{\pi}{2} = -\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}. \end{aligned}$$

Problem 3.31 Transform the following vectors into spherical coordinates and then evaluate them at the indicated points:

(a) $\mathbf{A} = \hat{\mathbf{x}}y^2 + \hat{\mathbf{y}}xz + \hat{\mathbf{z}}4$ at $P_1(1, -1, 2)$,

(b) $\mathbf{B} = \hat{\mathbf{y}}(x^2 + y^2 + z^2) - \hat{\mathbf{z}}(x^2 + y^2)$ at $P_2(-1, 0, 2)$,

(c) $\mathbf{C} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi + \hat{\mathbf{z}}\cos\phi\sin\phi$ at $P_3(2, \pi/4, 2)$, and

(d) $\mathbf{D} = \hat{\mathbf{x}}y^2/(x^2 + y^2) - \hat{\mathbf{y}}x^2/(x^2 + y^2) + \hat{\mathbf{z}}4$ at $P_4(1, -1, 2)$.

Solution: From Table 3-2:

(a)

$$\begin{aligned} \mathbf{A} &= (\hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)(R\sin\theta\sin\phi)^2 \\ &\quad + (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)(R\sin\theta\cos\phi)(R\cos\theta) \\ &\quad + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)4 \\ &= \hat{\mathbf{R}}(R^2\sin^2\theta\sin\phi\cos\phi(\sin\theta\sin\phi + \cos\theta) + 4\cos\theta) \\ &\quad + \hat{\boldsymbol{\theta}}(R^2\sin\theta\cos\theta\sin\phi\cos\phi(\sin\theta\sin\phi + \cos\theta) - 4\sin\theta) \\ &\quad + \hat{\boldsymbol{\phi}}R^2\sin\theta(\cos\theta\cos^2\phi - \sin\theta\sin^3\phi), \\ P_1 &= \left(\sqrt{1^2 + (-1)^2 + 2^2}, \tan^{-1}\left(\sqrt{1^2 + (-1)^2}/2\right), \tan^{-1}(-1/1)\right) \\ &= (\sqrt{6}, 35.3^\circ, -45^\circ), \\ \mathbf{A}(P_1) &\approx \hat{\mathbf{R}}2.856 - \hat{\boldsymbol{\theta}}2.888 + \hat{\boldsymbol{\phi}}2.123. \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{B} &= (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)R^2 - (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)R^2\sin^2\theta \\ &= \hat{\mathbf{R}}R^2\sin\theta(\sin\phi - \sin\theta\cos\theta) + \hat{\boldsymbol{\theta}}R^2(\cos\theta\sin\phi + \sin^3\theta) + \hat{\boldsymbol{\phi}}R^2\cos\phi, \\ P_2 &= \left(\sqrt{(-1)^2 + 0^2 + 2^2}, \tan^{-1}\left(\sqrt{(-1)^2 + 0^2}/2\right), \tan^{-1}(0/(-1))\right) \\ &= (\sqrt{5}, 26.6^\circ, 180^\circ), \end{aligned}$$

$$\mathbf{B}(P_2) \approx -\hat{\mathbf{R}}0.896 + \hat{\boldsymbol{\theta}}0.449 - \hat{\boldsymbol{\phi}}5.$$

(c)

$$\begin{aligned}
\mathbf{C} &= (\hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta)\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\cos\phi\sin\phi \\
&= \hat{\mathbf{R}}\cos\phi(\sin\theta + \cos\theta\sin\phi) + \hat{\boldsymbol{\theta}}\cos\phi(\cos\theta - \sin\theta\sin\phi) - \hat{\boldsymbol{\phi}}\sin\phi, \\
P_3 &= \left(\sqrt{2^2+2^2}, \tan^{-1}(2/2), \pi/4\right) = (2\sqrt{2}, 45^\circ, 45^\circ), \\
\mathbf{C}(P_3) &\approx \hat{\mathbf{R}}0.854 + \hat{\boldsymbol{\theta}}0.146 - \hat{\boldsymbol{\phi}}0.707.
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{D} = {} & (\hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)\,\frac{R^2\sin^2\theta\sin^2\phi}{R^2\sin^2\theta\sin^2\phi + R^2\sin^2\theta\cos^2\phi} \\
& - (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)\,\frac{R^2\sin^2\theta\cos^2\phi}{R^2\sin^2\theta\sin^2\phi + R^2\sin^2\theta\cos^2\phi} \\
& + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)4 \\
= {} & \hat{\mathbf{R}}(\sin\theta\cos\phi\sin^2\phi - \sin\theta\sin\phi\cos^2\phi + 4\cos\theta) \\
& + \hat{\boldsymbol{\theta}}(\cos\theta\cos\phi\sin^2\phi - \cos\theta\sin\phi\cos^2\phi - 4\sin\theta) \\
& - \hat{\boldsymbol{\phi}}(\cos^3\phi + \sin^3\phi),
\end{aligned}$$

$$\begin{aligned}
P_4(1,-1,2) &= P_4\left[\sqrt{1+1+4}, \tan^{-1}(\sqrt{1+1}/2), \tan^{-1}(-1/1)\right] \\
&= P_4(\sqrt{6}, 35.26^\circ, -45^\circ),
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}(P_4) = {} & \hat{\mathbf{R}}(\sin 35.26^\circ\cos 45^\circ\sin^2 45^\circ - \sin 35.26^\circ\sin(-45^\circ)\cos^2 45^\circ + 4\cos 35.26^\circ) \\
& + \hat{\boldsymbol{\theta}}(\cos 35.26^\circ\cos 45^\circ\sin^2 45^\circ - \cos 35.26^\circ\sin(-45^\circ)\cos^2 45^\circ - 4\sin 35.26^\circ) \\
& - \hat{\boldsymbol{\phi}}(\cos^3 45^\circ + \sin^3 45^\circ) \\
= {} & \hat{\mathbf{R}}3.67 - \hat{\boldsymbol{\theta}}1.73 - \hat{\boldsymbol{\phi}}0.707.
\end{aligned}$$

Sections 3-4 to 3-7: Gradient, Divergence, and Curl Operators

Problem 3.32 Find the gradient of the following scalar functions:

(a) $T = 3/(x^2+z^2)$,

(b) $V = xy^2z^4$,

(c) $U = z\cos\phi/(1+r^2)$,
(d) $W = e^{-R}\sin\theta$,
(e) $S = 4x^2e^{-z} + y^3$,
(f) $N = r^2\cos^2\phi$,
(g) $M = R\cos\theta\sin\phi$.

Solution:

(a) From Eq. (3.72),

$$\nabla T = -\hat{\mathbf{x}}\frac{6x}{(x^2+z^2)^2} - \hat{\mathbf{z}}\frac{6z}{(x^2+z^2)^2}\,.$$

(b) From Eq. (3.72),

$$\nabla V = \hat{\mathbf{x}}y^2z^4 + \hat{\mathbf{y}}2xyz^4 + \hat{\mathbf{z}}4xy^2z^3.$$

(c) From Eq. (3.82),

$$\nabla U = -\hat{\mathbf{r}}\frac{2rz\cos\phi}{(1+r^2)^2} - \hat{\boldsymbol{\phi}}\frac{z\sin\phi}{r(1+r^2)} + \hat{\mathbf{z}}\frac{\cos\phi}{1+r^2}\,.$$

(d) From Eq. (3.83),

$$\nabla W = -\hat{\mathbf{R}}e^{-R}\sin\theta + \hat{\boldsymbol{\theta}}(e^{-R}/R)\cos\theta.$$

(e) From Eq. (3.72),

$$S = 4x^2e^{-z} + y^3,$$
$$\nabla S = \hat{\mathbf{x}}\frac{\partial S}{\partial x} + \hat{\mathbf{y}}\frac{\partial S}{\partial y} + \hat{\mathbf{z}}\frac{\partial S}{\partial z} = \hat{\mathbf{x}}8xe^{-z} + \hat{\mathbf{y}}3y^2 - \hat{\mathbf{z}}4x^2e^{-z}.$$

(f) From Eq. (3.82),

$$N = r^2\cos^2\phi,$$
$$\nabla N = \hat{\mathbf{r}}\frac{\partial N}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial N}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial N}{\partial z} = \hat{\mathbf{r}}2r\cos^2\phi - \hat{\boldsymbol{\phi}}2r\sin\phi\cos\phi.$$

(g) From Eq. (3.83),

$$M = R\cos\theta\sin\phi,$$
$$\nabla M = \hat{\mathbf{R}}\frac{\partial M}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial M}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial M}{\partial \phi} = \hat{\mathbf{R}}\cos\theta\sin\phi - \hat{\boldsymbol{\theta}}\sin\theta\sin\phi + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{\tan\theta}\,.$$

Problem 3.33 The gradient of a scalar function T is given by

$$\nabla T = \hat{\mathbf{z}} e^{-3z}.$$

If $T = 10$ at $z = 0$, find $T(z)$.

Solution:

$$\nabla T = \hat{\mathbf{z}} e^{-3z}.$$

By choosing P_1 at $z = 0$ and P_2 at any point z, (3.76) becomes

$$\begin{aligned} T(z) - T(0) &= \int_0^z \nabla T \cdot d\mathbf{l}' = \int_0^z \hat{\mathbf{z}} e^{-3z'} \cdot (\hat{\mathbf{x}}\,dx' + \hat{\mathbf{y}}\,dy' + \hat{\mathbf{z}}\,dz') \\ &= \int_0^z e^{-3z'}\,dz' = -\frac{-e^{-3z'}}{3}\bigg|_0^z = \frac{1}{3}(1 - e^{-3z}). \end{aligned}$$

Hence,

$$T(z) = T(0) + \frac{1}{3}(1 - e^{-3z}) = 10 + \frac{1}{3}(1 - e^{-3z}).$$

Problem 3.34 Follow a procedure similar to that leading to Eq. (3.82) to derive the expression given by Eq. (3.83) for ∇ in spherical coordinates.

Solution: From the chain rule and Table 3-2,

$$\begin{aligned} \nabla T &= \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \\ &= \hat{\mathbf{x}}\left(\frac{\partial T}{\partial R}\frac{\partial R}{\partial x} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x}\right) \\ &\quad + \hat{\mathbf{y}}\left(\frac{\partial T}{\partial R}\frac{\partial R}{\partial y} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial y}\right) \\ &\quad + \hat{\mathbf{z}}\left(\frac{\partial T}{\partial R}\frac{\partial R}{\partial z} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial z}\right) \\ &= \hat{\mathbf{x}}\left(\frac{\partial T}{\partial R}\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} + \frac{\partial T}{\partial \theta}\frac{\partial}{\partial x}\tan^{-1}\left(\sqrt{x^2+y^2}/z\right) + \frac{\partial T}{\partial \phi}\frac{\partial}{\partial x}\tan^{-1}(y/x)\right) \\ &\quad + \hat{\mathbf{y}}\left(\frac{\partial T}{\partial R}\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2} + \frac{\partial T}{\partial \theta}\frac{\partial}{\partial y}\tan^{-1}\left(\sqrt{x^2+y^2}/z\right) + \frac{\partial T}{\partial \phi}\frac{\partial}{\partial y}\tan^{-1}(y/x)\right) \\ &\quad + \hat{\mathbf{z}}\left(\frac{\partial T}{\partial R}\frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2} + \frac{\partial T}{\partial \theta}\frac{\partial}{\partial z}\tan^{-1}\left(\sqrt{x^2+y^2}/z\right) + \frac{\partial T}{\partial \phi}\frac{\partial}{\partial z}\tan^{-1}(y/x)\right) \end{aligned}$$

$$
\begin{aligned}
&= \hat{\mathbf{x}}\left(\frac{\partial T}{\partial R}\frac{x}{\sqrt{x^2+y^2+z^2}}+\frac{\partial T}{\partial \theta}\frac{z}{x^2+y^2+z^2}\frac{x}{\sqrt{x^2+y^2}}+\frac{\partial T}{\partial \phi}\frac{-y}{x^2+y^2}\right)\\
&\quad +\hat{\mathbf{y}}\left(\frac{\partial T}{\partial R}\frac{y}{\sqrt{x^2+y^2+z^2}}+\frac{\partial T}{\partial \theta}\frac{z}{x^2+y^2+z^2}\frac{y}{\sqrt{x^2+y^2}}+\frac{\partial T}{\partial \phi}\frac{x}{x^2+y^2}\right)\\
&\quad +\hat{\mathbf{z}}\left(\frac{\partial T}{\partial R}\frac{z}{\sqrt{x^2+y^2+z^2}}+\frac{\partial T}{\partial \theta}\frac{-1}{x^2+y^2+z^2}\sqrt{x^2+y^2}+\frac{\partial T}{\partial \phi}0\right)\\
&= \hat{\mathbf{x}}\left(\frac{\partial T}{\partial R}\frac{R\sin\theta\cos\phi}{R}+\frac{\partial T}{\partial \theta}\frac{R\cos\theta}{R^2}\frac{R\sin\theta\cos\phi}{R\sin\theta}+\frac{\partial T}{\partial \phi}\frac{-R\sin\theta\sin\phi}{R^2\sin^2\theta}\right)\\
&\quad +\hat{\mathbf{y}}\left(\frac{\partial T}{\partial R}\frac{R\sin\theta\sin\phi}{R}+\frac{\partial T}{\partial \theta}\frac{R\cos\theta}{R^2}\frac{R\sin\theta\sin\phi}{R\sin\theta}+\frac{\partial T}{\partial \phi}\frac{R\sin\theta\cos\phi}{R^2\sin^2\theta}\right)\\
&\quad +\hat{\mathbf{z}}\left(\frac{\partial T}{\partial R}\frac{R\cos\theta}{R}+\frac{\partial T}{\partial \theta}\frac{-R\sin\theta}{R^2}\right)\\
&= \hat{\mathbf{x}}\left(\frac{\partial T}{\partial R}\sin\theta\cos\phi+\frac{\partial T}{\partial \theta}\frac{\cos\theta\cos\phi}{R}+\frac{\partial T}{\partial \phi}\frac{-\sin\phi}{R\sin\theta}\right)\\
&\quad +\hat{\mathbf{y}}\left(\frac{\partial T}{\partial R}\sin\theta\sin\phi+\frac{\partial T}{\partial \theta}\frac{\cos\theta\sin\phi}{R}+\frac{\partial T}{\partial \phi}\frac{\cos\phi}{R\sin\theta}\right)\\
&\quad +\hat{\mathbf{z}}\left(\frac{\partial T}{\partial R}\cos\theta+\frac{\partial T}{\partial \theta}\frac{-\sin\theta}{R}\right)\\
&= (\hat{\mathbf{x}}\sin\theta\cos\phi+\hat{\mathbf{y}}\sin\theta\sin\phi+\hat{\mathbf{z}}\cos\theta)\,\frac{\partial T}{\partial R}\\
&\quad +(\hat{\mathbf{x}}\cos\theta\cos\phi+\hat{\mathbf{y}}\cos\theta\sin\phi-\hat{\mathbf{z}}\sin\theta)\,\frac{1}{R}\frac{\partial T}{\partial \theta}\\
&\quad +(-\hat{\mathbf{x}}\sin\phi+\hat{\mathbf{y}}\cos\phi)\,\frac{1}{R\sin\theta}\frac{\partial T}{\partial \phi}\\
&= \hat{\mathbf{R}}\frac{\partial T}{\partial R}+\hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial T}{\partial \theta}+\hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial T}{\partial \phi}\,,
\end{aligned}
$$

which is Eq. (3.83).

Problem 3.35 For the scalar function $V = xy^2 - z^2$, determine its directional derivative along the direction of vector $\mathbf{A} = (\hat{\mathbf{x}} - \hat{\mathbf{y}}z)$ and then evaluate it at $P(1,-1,4)$.

Solution: The directional derivative is given by Eq. (3.75) as $dV/dl = \nabla V \cdot \hat{\mathbf{a}}_l$, where the unit vector in the direction of $\mathbf{A}$ is given by Eq. (3.2):

$$
\hat{\mathbf{a}}_l = \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}z}{\sqrt{1+z^2}}\,,
$$

and the gradient of V in Cartesian coordinates is given by Eq. (3.72):

$$\nabla V = \hat{\mathbf{x}} y^2 + \hat{\mathbf{y}} 2xy - \hat{\mathbf{z}} 2z.$$

Therefore, by Eq. (3.75),

$$\frac{dV}{dl} = \frac{y^2 - 2xyz}{\sqrt{1+z^2}}\,.$$

At $P(1,-1,4)$,

$$\left(\frac{dV}{dl}\right)\bigg|_{(1,-1,4)} = \frac{9}{\sqrt{17}} = 2.18\,.$$

Problem 3.36 For the scalar function $T = \frac{1}{2} e^{-r/5} \cos\phi$, determine its directional derivative along the radial direction $\hat{\mathbf{r}}$ and then evaluate it at $P(2,\pi/4,3)$.

Solution:

$$\begin{aligned}
T &= \frac{1}{2} e^{-r/5} \cos\phi, \\
\nabla T &= \hat{\mathbf{r}} \frac{\partial T}{\partial r} + \hat{\boldsymbol{\phi}} \frac{1}{r} \frac{\partial T}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial T}{\partial z} = -\hat{\mathbf{r}} \frac{e^{-r/5} \cos\phi}{10} - \hat{\boldsymbol{\phi}} \frac{e^{-r/5} \sin\phi}{2r}, \\
\frac{dT}{dl} &= \nabla T \cdot \hat{\mathbf{r}} = -\frac{e^{-r/5} \cos\phi}{10}, \\
\frac{dT}{dl}\bigg|_{(2,\pi/4,3)} &= -\frac{e^{-2/5} \cos\frac{\pi}{4}}{10} = -4.74 \times 10^{-2}.
\end{aligned}$$

Problem 3.37 For the scalar function $U = \frac{1}{R} \sin^2\theta$, determine its directional derivative along the range direction $\hat{\mathbf{R}}$ and then evaluate it at $P(5,\pi/4,\pi/2)$.

Solution:

$$\begin{aligned}
U &= \frac{1}{R} \sin^2\theta, \\
\nabla U &= \hat{\mathbf{R}} \frac{\partial U}{\partial R} + \hat{\boldsymbol{\theta}} \frac{1}{R} \frac{\partial U}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{R\sin\theta} \frac{\partial U}{\partial \phi} = -\hat{\mathbf{R}} \frac{\sin^2\theta}{R^2} - \hat{\boldsymbol{\theta}} \frac{2\sin\theta\cos\theta}{R}, \\
\frac{dU}{dl} &= \nabla U \cdot \hat{\mathbf{R}} = -\frac{\sin^2\theta}{R^2}, \\
\frac{dU}{dl}\bigg|_{(5,\pi/4,\pi/2)} &= -\frac{\sin^2(\pi/4)}{25} = -0.02\,.
\end{aligned}$$

Problem 3.38 Vector field **E** is characterized by the following properties: (a) **E** points along $\hat{\mathbf{R}}$, (b) the magnitude of **E** is a function of only the distance from the origin, (c) **E** vanishes at the origin, and (d) $\nabla \cdot \mathbf{E} = 12$, everywhere. Find an expression for **E** that satisfies these properties.

Solution: According to properties (a) and (b), **E** must have the form

$$\mathbf{E} = \hat{\mathbf{R}} E_R$$

where E_R is a function of R only.

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 E_R) = 12, \\
\frac{\partial}{\partial R} (R^2 E_R) &= 12R^2, \\
\int_0^R \frac{\partial}{\partial R} (R^2 E_R) \, dR &= \int_0^R 12R^2 \, dR, \\
R^2 E_R \big|_0^R &= \left. \frac{12R^3}{3} \right|_0^R, \\
R^2 E_R &= 4R^3.
\end{aligned}$$

Hence,

$$E_R = 4R,$$

and

$$\mathbf{E} = \hat{\mathbf{R}} 4R.$$

Problem 3.39 For the vector field $\mathbf{E} = \hat{\mathbf{x}} xz - \hat{\mathbf{y}} yz^2 - \hat{\mathbf{z}} xy$, verify the divergence theorem by computing:

(a) the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes, and

(b) the integral of $\nabla \cdot \mathbf{E}$ over the cube's volume.

Solution:

(a) For a cube, the closed surface integral has 6 sides:

$$\oint \mathbf{E} \cdot d\mathbf{s} = F_{\text{top}} + F_{\text{bottom}} + F_{\text{right}} + F_{\text{left}} + F_{\text{front}} + F_{\text{back}},$$

$$
\begin{aligned}
F_{\text{top}} &= \int_{x=-1}^{1}\int_{y=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{z=1} \cdot (\hat{\mathbf{z}}\,dy\,dx) \\
&= -\int_{x=-1}^{1}\int_{y=-1}^{1} xy\,dy\,dx = \left(\left(\frac{x^2y^2}{4}\right)\Bigg|_{y=-1}^{1}\right)\Bigg|_{x=-1}^{1} = 0, \\
F_{\text{bottom}} &= \int_{x=-1}^{1}\int_{y=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{z=-1} \cdot (-\hat{\mathbf{z}}\,dy\,dx) \\
&= \int_{x=-1}^{1}\int_{y=-1}^{1} xy\,dy\,dx = \left(\left(\frac{x^2y^2}{4}\right)\Bigg|_{y=-1}^{1}\right)\Bigg|_{x=-1}^{1} = 0, \\
F_{\text{right}} &= \int_{x=-1}^{1}\int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{y=1} \cdot (\hat{\mathbf{y}}\,dz\,dx) \\
&= -\int_{x=-1}^{1}\int_{z=-1}^{1} z^2\,dz\,dx = -\left(\left(\frac{xz^3}{3}\right)\Bigg|_{z=-1}^{1}\right)\Bigg|_{x=-1}^{1} = \frac{-4}{3}, \\
F_{\text{left}} &= \int_{x=-1}^{1}\int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{y=-1} \cdot (-\hat{\mathbf{y}}\,dz\,dx) \\
&= -\int_{x=-1}^{1}\int_{z=-1}^{1} z^2\,dz\,dx = -\left(\left(\frac{xz^3}{3}\right)\Bigg|_{z=-1}^{1}\right)\Bigg|_{x=-1}^{1} = \frac{-4}{3}, \\
F_{\text{front}} &= \int_{y=-1}^{1}\int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{x=1} \cdot (\hat{\mathbf{x}}\,dz\,dy) \\
&= \int_{y=-1}^{1}\int_{z=-1}^{1} z\,dz\,dy = \left(\left(\frac{yz^2}{2}\right)\Bigg|_{z=-1}^{1}\right)\Bigg|_{y=-1}^{1} = 0, \\
F_{\text{back}} &= \int_{y=-1}^{1}\int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\big|_{x=-1} \cdot (-\hat{\mathbf{x}}\,dz\,dy) \\
&= \int_{y=-1}^{1}\int_{z=-1}^{1} z\,dz\,dy = \left(\left(\frac{yz^2}{2}\right)\Bigg|_{z=-1}^{1}\right)\Bigg|_{y=-1}^{1} = 0, \\
\oint \mathbf{E}\cdot d\mathbf{s} &= 0 + 0 + \frac{-4}{3} + \frac{-4}{3} + 0 + 0 = \frac{-8}{3}\,.
\end{aligned}
$$

(b)

$$\begin{aligned}\iiint \nabla\cdot\mathbf{E}\,dv &= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1} \nabla\cdot(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy)\,dz\,dy\,dx \\ &= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1} (z - z^2)\,dz\,dy\,dx \\ &= \left(\left(\left(xy\left(\frac{z^2}{2} - \frac{z^3}{3}\right)\right)\Bigg|_{z=-1}^{1}\right)\Bigg|_{y=-1}^{1}\right)\Bigg|_{x=-1}^{1} = \frac{-8}{3}.\end{aligned}$$

Problem 3.40 For the vector field $\mathbf{E} = \hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z$, verify the divergence theorem for the cylindrical region enclosed by $r = 2$, $z = 0$, and $z = 4$.

Solution:

$$\begin{aligned}\oint \mathbf{E}\cdot d\mathbf{s} &= \int_{r=0}^{2}\int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z)\cdot(-\hat{\mathbf{z}}r\,dr\,d\phi)\right)\big|_{z=0} \\ &\quad + \int_{\phi=0}^{2\pi}\int_{z=0}^{4} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z)\cdot(\hat{\mathbf{r}}r\,d\phi\,dz)\right)\big|_{r=2} \\ &\quad + \int_{r=0}^{2}\int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)\right)\big|_{z=4} \\ &= 0 + \int_{\phi=0}^{2\pi}\int_{z=0}^{4} 10e^{-2}2\,d\phi\,dz + \int_{r=0}^{2}\int_{\phi=0}^{2\pi} -12r\,dr\,d\phi \\ &= 160\pi e^{-2} - 48\pi \approx -82.77, \\ \iiint \nabla\cdot\mathbf{E}\,d\mathcal{V} &= \int_{z=0}^{4}\int_{r=0}^{2}\int_{\phi=0}^{2\pi} \left(\frac{10e^{-r}(1-r)}{r} - 3\right) r\,d\phi\,dr\,dz \\ &= 8\pi\int_{r=0}^{2} (10e^{-r}(1-r) - 3r)\,dr \\ &= 8\pi\left(-10e^{-r} + 10e^{-r}(1+r) - \frac{3r^2}{2}\right)\Bigg|_{r=0}^{2} \\ &= 160\pi e^{-2} - 48\pi \approx -82.77.\end{aligned}$$

Problem 3.41 A vector field $\mathbf{D} = \hat{\mathbf{r}}r^3$ exists in the region between two concentric cylindrical surfaces defined by $r = 1$ and $r = 2$, with both cylinders extending between $z = 0$ and $z = 5$. Verify the divergence theorem by evaluating:

(a) $\oint_S \mathbf{D}\cdot d\mathbf{s}$,

(b) $\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V}$.

Solution:

(a)

$$\iint \mathbf{D} \cdot d\mathbf{s} = F_{\text{inner}} + F_{\text{outer}} + F_{\text{bottom}} + F_{\text{top}},$$

$$\begin{aligned}
F_{\text{inner}} &= \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left((\hat{\mathbf{r}} r^3) \cdot (-\hat{\mathbf{r}} r\, dz\, d\phi) \right) \Big|_{r=1} \\
&= \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left(-r^4\, dz\, d\phi \right) \Big|_{r=1} = -10\pi, \\
F_{\text{outer}} &= \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left((\hat{\mathbf{r}} r^3) \cdot (\hat{\mathbf{r}} r\, dz\, d\phi) \right) \Big|_{r=2} \\
&= \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left(r^4\, dz\, d\phi \right) \Big|_{r=2} = 160\pi, \\
F_{\text{bottom}} &= \int_{r=1}^{2} \int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}} r^3) \cdot (-\hat{\mathbf{z}} r\, d\phi\, dr) \right) \Big|_{z=0} = 0, \\
F_{\text{top}} &= \int_{r=1}^{2} \int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}} r^3) \cdot (\hat{\mathbf{z}} r\, d\phi\, dr) \right) \Big|_{z=5} = 0.
\end{aligned}$$

Therefore, $\iint \mathbf{D} \cdot d\mathbf{s} = 150\pi$.

(b) From the back cover, $\nabla \cdot \mathbf{D} = (1/r)(\partial/\partial r)(r r^3) = 4r^2$. Therefore,

$$\iiint \nabla \cdot \mathbf{D}\, d\mathcal{V} = \int_{z=0}^{5} \int_{\phi=0}^{2\pi} \int_{r=1}^{2} 4r^2 r\, dr\, d\phi\, dz = \left(\left(\left(r^4 \right) \Big|_{r=1}^{2} \right) \Big|_{\phi=0}^{2\pi} \right) \Bigg|_{z=0}^{5} = 150\pi.$$

Problem 3.42 For the vector field $\mathbf{D} = \hat{\mathbf{R}} 3R^2$, evaluate both sides of the divergence theorem for the region enclosed between the spherical shells defined by $R = 1$ and $R = 2$.

Solution: The divergence theorem is given by Eq. (3.98). Evaluating the left hand side:

$$\begin{aligned}
\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=1}^{2} \left(\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 (3R^2)) \right) R^2 \sin\theta\, dR\, d\theta\, d\phi \\
&= 2\pi (-\cos\theta) \big|_{\theta=0}^{\pi} \left(3R^4 \right) \big|_{R=1}^{2} = 180\pi.
\end{aligned}$$

The right hand side evaluates to

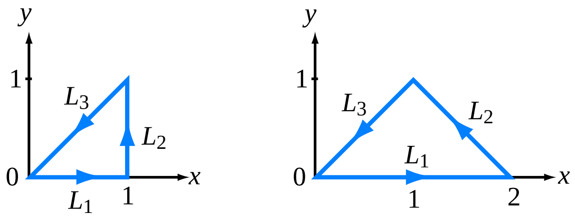
$$\begin{aligned}\oint_S \mathbf{D}\cdot d\mathbf{s} &= \left(\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}(\hat{\mathbf{R}}3R^2)\cdot(-\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi)\right)\Bigg|_{R=1} \\ &\quad+\left(\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}(\hat{\mathbf{R}}3R^2)\cdot(\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi)\right)\Bigg|_{R=2} \\ &= -2\pi\int_{\theta=0}^{\pi}3\sin\theta\,d\theta+2\pi\int_{\theta=0}^{\pi}48\sin\theta\,d\theta = 180\pi.\end{aligned}$$

Problem 3.43 For the vector field $\mathbf{E}=\hat{\mathbf{x}}xy-\hat{\mathbf{y}}(x^2+2y^2)$, calculate

(a) $\oint_C \mathbf{E}\cdot d\mathbf{l}$ around the triangular contour shown in Fig. P3.43(a), and

(b) $\int_S (\nabla\times\mathbf{E})\cdot d\mathbf{s}$ over the area of the triangle.

Solution: In addition to the independent condition that $z=0$, the three lines of the triangle are represented by the equations $y=0$, $x=1$, and $y=x$, respectively.



(a) (b)

Figure P3.43: Contours for (a) Problem 3.43 and (b) Problem 3.44.

(a)

$$\oint \mathbf{E}\cdot d\mathbf{l} = L_1+L_2+L_3,$$

$$\begin{aligned}L_1 &= \int(\hat{\mathbf{x}}xy-\hat{\mathbf{y}}(x^2+2y^2))\cdot(\hat{\mathbf{x}}\,dx+\hat{\mathbf{y}}\,dy+\hat{\mathbf{z}}\,dz) \\ &= \int_{x=0}^{1}(xy)\big|_{y=0,z=0}\,dx-\int_{y=0}^{0}\left(x^2+2y^2\right)\big|_{z=0}\,dy+\int_{z=0}^{0}(0)\big|_{y=0}\,dz=0,\end{aligned}$$

$$
\begin{aligned}
L_2 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz) \\
&= \int_{x=1}^{1} (xy)\big|_{z=0}\,dx - \int_{y=0}^{1} \left(x^2+2y^2\right)\big|_{x=1,z=0}\,dy + \int_{z=0}^{0} (0)\big|_{x=1}\,dz \\
&= 0 - \left(y + \frac{2y^3}{3}\right)\bigg|_{y=0}^{1} + 0 = \frac{-5}{3}\,, \\
L_3 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz) \\
&= \int_{x=1}^{0} (xy)\big|_{y=x,\ \ z=0}\,dx - \int_{y=1}^{0} \left(x^2+2y^2\right)\big|_{x=y,\ \ z=0}\,dy + \int_{z=0}^{0} (0)\big|_{y=x}\,dz \\
&= \left(\frac{x^3}{3}\right)\bigg|_{x=1}^{0} - \left(y^3\right)\big|_{y=1}^{0} + 0 = \frac{2}{3}\,.
\end{aligned}
$$

Therefore,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 - \frac{5}{3} + \frac{2}{3} = -1.$$

(b) From Eq. (3.105), $\nabla \times \mathbf{E} = -\hat{\mathbf{z}}3x$, so that

$$
\begin{aligned}
\iint \nabla \times \mathbf{E} \cdot d\mathbf{s} &= \int_{x=0}^{1}\int_{y=0}^{x} ((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}\,dy\,dx))\big|_{z=0} \\
&= -\int_{x=0}^{1}\int_{y=0}^{x} 3x\,dy\,dx = -\int_{x=0}^{1} 3x(x-0)\,dx = -\left(x^3\right)\big|_{0}^{1} = -1.
\end{aligned}
$$

Problem 3.44 Repeat Problem 3.43 for the contour shown in Fig. P3.43(b).

Solution: In addition to the independent condition that $z = 0$, the three lines of the triangle are represented by the equations $y = 0$, $y = 2 - x$, and $y = x$, respectively.

(a)

$$\oint \mathbf{E} \cdot d\mathbf{l} = L_1 + L_2 + L_3,$$

$$
\begin{aligned}
L_1 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz) \\
&= \int_{x=0}^{2} (xy)\big|_{y=0,z=0}\,dx - \int_{y=0}^{0} \left(x^2+2y^2\right)\big|_{z=0}\,dy + \int_{z=0}^{0} (0)\big|_{y=0}\,dz = 0, \\
L_2 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz) \\
&= \int_{x=2}^{1} (xy)\big|_{z=0,y=2-x}\,dx - \int_{y=0}^{1} \left(x^2+2y^2\right)\big|_{x=2-y,z=0}\,dy + \int_{z=0}^{0} (0)\big|_{y=2-x}\,dz \\
&= \left(x^2 - \frac{x^3}{3}\right)\bigg|_{x=2}^{1} - \left(4y - 2y^2 + y^3\right)\big|_{y=0}^{1} + 0 = \frac{-11}{3}\,,
\end{aligned}
$$

$$
\begin{aligned}
L_3 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2+2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz) \\
&= \int_{x=1}^{0} (xy)\big|_{y=x,z=0}\,dx - \int_{y=1}^{0} \left(x^2+2y^2\right)\big|_{x=y,z=0}\,dy + \int_{z=0}^{0} (0)\big|_{y=x}\,dz \\
&= \left(\frac{x^3}{3}\right)\Bigg|_{x=1}^{0} - \left(y^3\right)\big|_{y=1}^{0} + 0 = \frac{2}{3}\,.
\end{aligned}
$$

Therefore,

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0 - \frac{11}{3} + \frac{2}{3} = -3.$$

(b) From Eq. (3.105), $\nabla\times\mathbf{E} = -\hat{\mathbf{z}}3x$, so that

$$
\begin{aligned}
\iint \nabla\times\mathbf{E}\cdot d\mathbf{s} &= \int_{x=0}^{1}\int_{y=0}^{x} \left((-\hat{\mathbf{z}}3x)\cdot(\hat{\mathbf{z}}\,dy\,dx)\right)\big|_{z=0} \\
&\quad + \int_{x=1}^{2}\int_{y=0}^{2-x} \left((-\hat{\mathbf{z}}3x)\cdot(\hat{\mathbf{z}}\,dy\,dx)\right)\big|_{z=0} \\
&= -\int_{x=0}^{1}\int_{y=0}^{x} 3x\,dy\,dx - \int_{x=1}^{2}\int_{y=0}^{2-x} 3x\,dy\,dx \\
&= -\int_{x=0}^{1} 3x(x-0)\,dx - \int_{x=1}^{2} 3x((2-x)-0)\,dx \\
&= -\left(x^3\right)\big|_{0}^{1} - \left(3x^2-x^3\right)\big|_{x=1}^{2} = -3.
\end{aligned}
$$

Problem 3.45 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$ by evaluating:

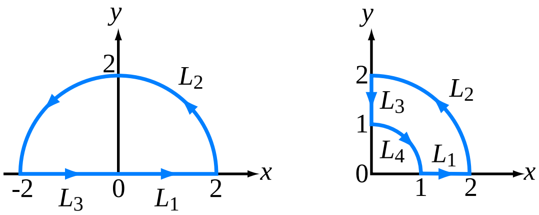
(a) $\oint_C \mathbf{B}\cdot d\mathbf{l}$ over the semicircular contour shown in Fig. P3.46(a), and

(b) $\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s}$ over the surface of the semicircle.

Solution:

(a)

$$
\begin{aligned}
\oint \mathbf{B}\cdot d\mathbf{l} &= \int_{L_1} \mathbf{B}\cdot d\mathbf{l} + \int_{L_2} \mathbf{B}\cdot d\mathbf{l} + \int_{L_3} \mathbf{B}\cdot d\mathbf{l}, \\
\mathbf{B}\cdot d\mathbf{l} &= (\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi, \\
\int_{L_1} \mathbf{B}\cdot d\mathbf{l} &= \left(\int_{r=0}^{2} r\cos\phi\,dr\right)\Bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\Bigg|_{z=0} \\
&= \left(\tfrac{1}{2}r^2\right)\big|_{r=0}^{2} + 0 = 2,
\end{aligned}
$$



(a) (b)

Figure P3.46: Contour paths for (a) Problem 3.45 and (b) Problem 3.46.

$$\int_{L_2} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=0}^{\pi} r\sin\phi\,d\phi\right)\bigg|_{r=2,\ \ z=0}$$

$$= 0 + (-2\cos\phi)\big|_{\phi=0}^{\pi} = 4,$$

$$\int_{L_3} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{0} r\cos\phi\,dr\right)\bigg|_{\phi=\pi, z=0} + \left(\int_{\phi=\pi}^{\pi} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(-\tfrac{1}{2}r^2\right)\big|_{r=2}^{0} + 0 = 2,$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = 2+4+2 = 8.$$

(b)

$$\nabla\times\mathbf{B} = \nabla\times(\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$$

$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}(\sin\phi)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi) - \frac{\partial}{\partial r}0\right)$$

$$+\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi)) - \frac{\partial}{\partial\phi}(r\cos\phi)\right)$$

$$= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + (r\sin\phi)) = \hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right),$$

$$\iint \nabla\times\mathbf{B}\cdot d\mathbf{s} = \int_{\phi=0}^{\pi}\int_{r=0}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)$$

$$= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\sin\phi(r+1)\,dr\,d\phi = \left(\left(-\cos\phi\left(\tfrac{1}{2}r^2+r\right)\right)\Big|_{r=0}^{2}\right)\bigg|_{\phi=0}^{\pi} = 8.$$

Problem 3.46 Repeat Problem 3.45 for the contour shown in Fig. P3.46(b).

Solution:

(a)

$$
\begin{aligned}
\oint \mathbf{B}\cdot d\mathbf{l} &= \int_{L_1}\mathbf{B}\cdot d\mathbf{l} + \int_{L_2}\mathbf{B}\cdot d\mathbf{l} + \int_{L_3}\mathbf{B}\cdot d\mathbf{l} + \int_{L_4}\mathbf{B}\cdot d\mathbf{l},\\
\mathbf{B}\cdot d\mathbf{l} &= (\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,\\
\int_{L_1}\mathbf{B}\cdot d\mathbf{l} &= \left(\int_{r=1}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}\\
&= \left(\tfrac{1}{2}r^2\right)\Big|_{r=1}^{2} + 0 = \frac{3}{2},\\
\int_{L_2}\mathbf{B}\cdot d\mathbf{l} &= \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=0}^{\pi/} r\sin\phi\,d\phi\right)\bigg|_{r=2,\ z=0}\\
&= 0 + (-2\cos\phi)\Big|_{\phi=0}^{\pi/2} = 2,\\
\int_{L_3}\mathbf{B}\cdot d\mathbf{l} &= \left(\int_{r=2}^{1} r\cos\phi\,dr\right)\bigg|_{\phi=\pi/2,\ z=0} + \left(\int_{\phi=\pi/2}^{\pi/2} r\sin\phi\,d\phi\right)\bigg|_{z=0} = 0,\\
\int_{L_4}\mathbf{B}\cdot d\mathbf{l} &= \left(\int_{r=1}^{1} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=\pi/2}^{0} r\sin\phi\,d\phi\right)\bigg|_{r=1,\ z=0}\\
&= 0 + (-\cos\phi)\Big|_{\phi=\pi/2}^{0} = -1,\\
\oint \mathbf{B}\cdot d\mathbf{l} &= \frac{3}{2} + 2 + 0 - 1 = \frac{5}{2}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
\nabla\times\mathbf{B} &= \nabla\times(\hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\\
&= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}(\sin\phi)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi) - \frac{\partial}{\partial r}0\right)\\
&\quad + \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi)) - \frac{\partial}{\partial\phi}(r\cos\phi)\right)\\
&= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + (r\sin\phi)) = \hat{\mathbf{z}}\sin\phi\left(1 + \frac{1}{r}\right),\\
\iint \nabla\times\mathbf{B}\cdot d\mathbf{s} &= \int_{\phi=0}^{\pi/2}\int_{r=1}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)\\
&= \int_{\phi=0}^{\pi/2}\int_{r=1}^{2}\sin\phi(r+1)\,dr\,d\phi\\
&= \left(\left(-\cos\phi\left(\tfrac{1}{2}r^2 + r\right)\right)\Big|_{r=1}^{2}\right)\bigg|_{\phi=0}^{\pi/2} = \frac{5}{2}.
\end{aligned}
$$

Problem 3.47 Verify Stokes's Theorem for the vector field $\mathbf{A} = \hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta$ by evaluating it on the hemisphere of unit radius.

Solution:

$$\mathbf{A} = \hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi.$$

Hence, $A_R = \cos\theta$, $A_\theta = 0$, $A_\phi = \sin\theta$.

$$\begin{aligned}\nabla \times \mathbf{A} &= \hat{\mathbf{R}}\,\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi \sin\theta)\right) - \hat{\boldsymbol{\theta}}\,\frac{1}{R}\,\frac{\partial}{\partial R}\,(RA_\phi) - \hat{\boldsymbol{\phi}}\,\frac{1}{R}\,\frac{\partial A_R}{\partial\theta} \\ &= \hat{\mathbf{R}}\,\frac{1}{R\sin\theta}\,\frac{\partial}{\partial\theta}\,(\sin^2\theta) - \hat{\boldsymbol{\theta}}\,\frac{1}{R}\,\frac{\partial}{\partial R}\,(R\sin\theta) - \hat{\boldsymbol{\phi}}\,\frac{1}{R}\,\frac{\partial}{\partial\theta}\,(\cos\theta) \\ &= \hat{\mathbf{R}}\,\frac{2\cos\theta}{R} - \hat{\boldsymbol{\theta}}\,\frac{\sin\theta}{R} + \hat{\boldsymbol{\phi}}\,\frac{\sin\theta}{R}.\end{aligned}$$

For the hemispherical surface, $d\mathbf{s} = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$.

$$\begin{aligned}&\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}(\nabla \times \mathbf{A})\cdot d\mathbf{s} \\ &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left(\frac{\hat{\mathbf{R}}2\cos\theta}{R} - \hat{\boldsymbol{\theta}}\,\frac{\sin\theta}{R} + \hat{\boldsymbol{\phi}}\,\frac{\sin\theta}{R}\right)\cdot \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi\bigg|_{R=1} \\ &= 4\pi R\left.\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2}\right|_{R=1} = 2\pi.\end{aligned}$$

The contour C is the circle in the x–y plane bounding the hemispherical surface.

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_{\phi=0}^{2\pi}(\hat{\mathbf{R}}\cos\theta + \hat{\boldsymbol{\phi}}\sin\theta)\cdot\hat{\boldsymbol{\phi}}R\,d\phi\bigg|_{\substack{\theta=\pi/2\\R=1}} = R\sin\theta\int_0^{2\pi}d\phi\bigg|_{\substack{\theta=\pi/2\\R=1}} = 2\pi.$$

Problem 3.48 Determine if each of the following vector fields is solenoidal, conservative, or both:

(a) $\mathbf{A} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y2xy$,
(b) $\mathbf{B} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z$,
(c) $\mathbf{C} = \hat{\mathbf{r}}(\sin\phi)/r^2 + \hat{\boldsymbol{\phi}}(\cos\phi)/r^2$,
(d) $\mathbf{D} = \hat{\mathbf{R}}/R$,
(e) $\mathbf{E} = \hat{\mathbf{r}}\left(3 - \frac{r}{1+r}\right) + \hat{\mathbf{z}}z$,
(f) $\mathbf{F} = (\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)/(x^2 + y^2)$,
(g) $\mathbf{G} = \hat{\mathbf{x}}(x^2 + z^2) - \hat{\mathbf{y}}(y^2 + x^2) - \hat{\mathbf{z}}(y^2 + z^2)$,
(h) $\mathbf{H} = \hat{\mathbf{R}}(Re^{-R})$.

Solution:

(a)

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= \nabla\cdot(\hat{\mathbf{x}}x^2-\hat{\mathbf{y}}2xy) = \frac{\partial}{\partial x}x^2-\frac{\partial}{\partial y}2xy = 2x-2x = 0,\\
\nabla\times\mathbf{A} &= \nabla\times(\hat{\mathbf{x}}x^2-\hat{\mathbf{y}}2xy)\\
&= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}0-\frac{\partial}{\partial z}(-2xy)\right)+\hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2)-\frac{\partial}{\partial x}0\right)+\hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-2xy)-\frac{\partial}{\partial y}(x^2)\right)\\
&= \hat{\mathbf{x}}0+\hat{\mathbf{y}}0-\hat{\mathbf{z}}(2y)\neq 0\,.
\end{aligned}$$

The field **A** is solenoidal but not conservative.

(b)

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= \nabla\cdot(\hat{\mathbf{x}}x^2-\hat{\mathbf{y}}y^2+\hat{\mathbf{z}}2z) = \frac{\partial}{\partial x}x^2-\frac{\partial}{\partial y}y^2+\frac{\partial}{\partial z}2z = 2x-2y+2\neq 0,\\
\nabla\times\mathbf{B} &= \nabla\times(\hat{\mathbf{x}}x^2-\hat{\mathbf{y}}y^2+\hat{\mathbf{z}}2z)\\
&= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(2z)-\frac{\partial}{\partial z}(-y^2)\right)+\hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2)-\frac{\partial}{\partial x}(2z)\right)+\hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-y^2)-\frac{\partial}{\partial y}(x^2)\right)\\
&= \hat{\mathbf{x}}0+\hat{\mathbf{y}}0+\hat{\mathbf{z}}0.
\end{aligned}$$

The field **B** is conservative but not solenoidal.

(c)

$$\begin{aligned}
\nabla\cdot\mathbf{C} &= \nabla\cdot\left(\hat{\mathbf{r}}\frac{\sin\phi}{r^2}+\hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\sin\phi}{r^2}\right)\right)+\frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{r^2}\right)+\frac{\partial}{\partial z}0\\
&= \frac{-\sin\phi}{r^3}+\frac{-\sin\phi}{r^3}+0 = \frac{-2\sin\phi}{r^3}\,,\\
\nabla\times\mathbf{C} &= \nabla\times\left(\hat{\mathbf{r}}\frac{\sin\phi}{r^2}+\hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\\
&= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0-\frac{\partial}{\partial z}\left(\frac{\cos\phi}{r^2}\right)\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}\left(\frac{\sin\phi}{r^2}\right)-\frac{\partial}{\partial r}0\right)\\
&\quad+\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\left(\frac{\cos\phi}{r^2}\right)\right)-\frac{\partial}{\partial\phi}\left(\frac{\sin\phi}{r^2}\right)\right)\\
&= \hat{\mathbf{r}}0+\hat{\boldsymbol{\phi}}0+\hat{\mathbf{z}}\frac{1}{r}\left(-\left(\frac{\cos\phi}{r^2}\right)-\left(\frac{\cos\phi}{r^2}\right)\right) = \hat{\mathbf{z}}\frac{-2\cos\phi}{r^3}\,.
\end{aligned}$$

The field **C** is neither solenoidal nor conservative.

(d)

$$\nabla\cdot\mathbf{D}=\nabla\cdot\left(\frac{\hat{\mathbf{R}}}{R}\right)=\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\left(\frac{1}{R}\right)\right)+\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(0\sin\theta)+\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}0=\frac{1}{R^2},$$

$$\begin{aligned}\nabla\times\mathbf{D}&=\nabla\times\left(\frac{\hat{\mathbf{R}}}{R}\right)\\&=\hat{\mathbf{R}}\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(0\sin\theta)-\frac{\partial}{\partial\phi}0\right)+\hat{\boldsymbol{\theta}}\frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right)-\frac{\partial}{\partial R}(R(0))\right)\\&\quad+\hat{\boldsymbol{\phi}}\frac{1}{R}\left(\frac{\partial}{\partial R}(R(0))-\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\right)=\hat{\mathbf{r}}0+\hat{\boldsymbol{\theta}}0+\hat{\boldsymbol{\phi}}0.\end{aligned}$$

The field **D** is conservative but not solenoidal.

(e)

$$\begin{aligned}\mathbf{E}&=\hat{\mathbf{r}}\left(3-\frac{r}{1+r}\right)+\hat{\mathbf{z}}z,\\\nabla\cdot\mathbf{E}&=\frac{1}{r}\frac{\partial}{\partial r}(rE_{\mathrm{r}})+\frac{1}{r}\frac{\partial E_\phi}{\partial\phi}+\frac{\partial E_z}{\partial z}\\&=\frac{1}{r}\frac{\partial}{\partial r}\left(3r-\frac{r^2}{1+r}\right)+1\\&=\frac{1}{r}\left[3-\frac{2r}{1+r}+\frac{r^2}{(1+r)^2}\right]+1\\&=\frac{1}{r}\left[\frac{3+3r^2+6r-2r-2r^2+r^2}{(1+r)^2}\right]+1=\frac{2r^2+4r+3}{r(1+r)^2}+1\neq0,\\\nabla\times\mathbf{E}&=\hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial E_z}{\partial\phi}-\frac{\partial E_\phi}{\partial z}\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}\right)+\hat{\mathbf{z}}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)-\frac{1}{r}\frac{\partial E_r}{\partial\phi}\right)=0.\end{aligned}$$

Hence, **E** is conservative, but not solenoidal.

(f)

$$\begin{aligned}\mathbf{F}&=\frac{\hat{\mathbf{x}}y+\hat{\mathbf{y}}x}{x^2+y^2}=\hat{\mathbf{x}}\frac{y}{x^2+y^2}+\hat{\mathbf{y}}\frac{x}{x^2+y^2},\\\nabla\cdot\mathbf{F}&=\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\\&=\frac{-2xy}{(x^2+y^2)^2}+\frac{-2xy}{(x^2+y^2)^2}\neq0,\end{aligned}$$

$$\begin{aligned}\nabla\times\mathbf{F}&=\hat{\mathbf{x}}(0-0)+\hat{\mathbf{y}}(0-0)+\hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]\\&=\hat{\mathbf{z}}\left(\frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2}-\frac{1}{x^2+y^2}+\frac{2y^2}{(x^2+y^2)^2}\right)\\&=\hat{\mathbf{z}}\,\frac{2(y^2-x^2)}{(x^2+y^2)^2}\neq 0.\end{aligned}$$

Hence, **F** is neither solenoidal nor conservative.

(g)

$$\begin{aligned}\mathbf{G}&=\hat{\mathbf{x}}(x^2+z^2)-\hat{\mathbf{y}}(y^2+x^2)-\hat{\mathbf{z}}(y^2+z^2),\\\nabla\cdot\mathbf{G}&=\frac{\partial}{\partial x}(x^2+z^2)-\frac{\partial}{\partial y}(y^2+x^2)-\frac{\partial}{\partial z}(y^2+z^2)\\&=2x-2y-2z\neq 0,\\\nabla\times\mathbf{G}&=\hat{\mathbf{x}}\left(-\frac{\partial}{\partial y}(y^2+z^2)+\frac{\partial}{\partial z}(y^2+x^2)\right)+\hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2+z^2)+\frac{\partial}{\partial x}(y^2+z^2)\right)\\&\quad+\hat{\mathbf{z}}\left(-\frac{\partial}{\partial x}(y^2+x^2)-\frac{\partial}{\partial y}(x^2+z^2)\right)\\&=-\hat{\mathbf{x}}\,2y+\hat{\mathbf{y}}\,2z-\hat{\mathbf{z}}\,2x\neq 0.\end{aligned}$$

Hence, **G** is neither solenoidal nor conservative.

(h)

$$\begin{aligned}\mathbf{H}&=\hat{\mathbf{R}}(Re^{-R}),\\\nabla\cdot\mathbf{H}&=\frac{1}{R^2}\frac{\partial}{\partial R}(R^3e^{-R})=\frac{1}{R^2}(3R^2e^{-R}-R^3e^{-R})=e^{-R}(3-R)\neq 0,\\\nabla\times\mathbf{H}&=0.\end{aligned}$$

Hence, **H** is conservative, but not solenoidal.

Problem 3.49 Find the Laplacian of the following scalar functions:

(a) $V=4xy^2z^3$,
(b) $V=xy+yz+zx$,
(c) $V=3/(x^2+y^2)$,
(d) $V=5e^{-r}\cos\phi$,
(e) $V=10e^{-R}\sin\theta$.

Solution:

(a) From Eq. (3.110), $\nabla^2(4xy^2z^3)=8xz^3+24xy^2z$.

(b) $\nabla^2(xy+yz+zx)=0$.

(c) From the inside back cover of the book,

$$\nabla^2\left(\frac{3}{x^2+y^2}\right)=\nabla^2(3r^{-2})=12r^{-4}=\frac{12}{(x^2+y^2)^2}\,.$$

(d)

$$\nabla^2(5e^{-r}\cos\phi)=5e^{-r}\cos\phi\left(1-\frac{1}{r}-\frac{1}{r^2}\right).$$

(e)

$$\nabla^2(10e^{-R}\sin\theta)=10e^{-R}\left(\sin\theta\left(1-\frac{2}{R}\right)+\frac{\cos^2\theta-\sin^2\theta}{R^2\sin\theta}\right).$$

Problem 3.50 Find a vector **G** whose magnitude is 4 and whose direction is perpendicular to both vectors **E** and **F**, where $\mathbf{E}=\hat{\mathbf{x}}+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2$ and $\mathbf{F}=\hat{\mathbf{y}}3-\hat{\mathbf{z}}6$.

Solution: The cross product of two vectors produces a third vector which is perpendicular to both of the original vectors. Two vectors exist that satisfy the stated conditions, one along $\mathbf{E}\times\mathbf{F}$ and another along the opposite direction. Hence,

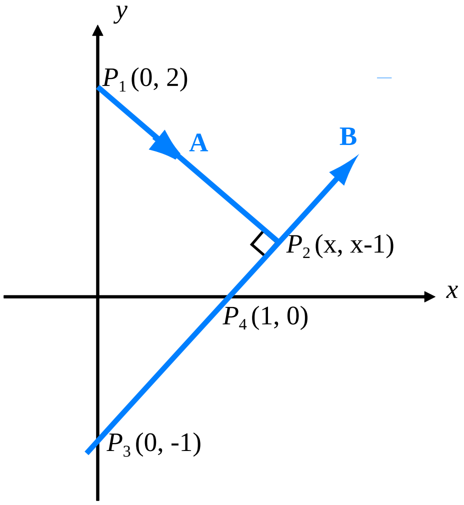
$$\begin{aligned}\mathbf{G}=\pm4\frac{\mathbf{E}\times\mathbf{F}}{|\mathbf{E}\times\mathbf{F}|}&=\pm4\frac{(\hat{\mathbf{x}}+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2)\times(\hat{\mathbf{y}}3-\hat{\mathbf{z}}6)}{|(\hat{\mathbf{x}}+\hat{\mathbf{y}}2-\hat{\mathbf{z}}2)\times(\hat{\mathbf{y}}3-\hat{\mathbf{z}}6)|}\\&=\pm4\frac{(-\hat{\mathbf{x}}6+\hat{\mathbf{y}}6+\hat{\mathbf{z}}3)}{\sqrt{36+36+9}}\\&=\pm\frac{4}{9}\,(-\hat{\mathbf{x}}6+\hat{\mathbf{y}}6+\hat{\mathbf{z}}3)=\pm\left(-\hat{\mathbf{x}}\frac{8}{3}+\hat{\mathbf{y}}\frac{8}{3}+\hat{\mathbf{z}}\frac{4}{3}\right).\end{aligned}$$

Problem 3.51 A given line is described by the equation:

$$y=x-1.$$

Vector **A** starts at point $P_1(0,2)$ and ends at point P_2 on the line such that **A** is orthogonal to the line. Find an expression for **A**.

Solution: We first plot the given line.



Next we find a vector **B** which connects point $P_3(0,1)$ to point $P_4(1,0)$, both of which are on the line. Hence,

$$\mathbf{B} = \hat{\mathbf{x}}(1-0)+\hat{\mathbf{y}}(0+1) = \hat{\mathbf{x}}+\hat{\mathbf{y}}.$$

Vector **A** starts at $P_1(0,2)$ and ends on the line at P_2. If the x-coordinate of P_2 is x, then its y-coordinate has to be $y = x-1$, per the equation for the line. Thus, P_2 is at $(x, x-1)$, and vector **A** is

$$\mathbf{A} = \hat{\mathbf{x}}(x-0)+\hat{\mathbf{y}}(x-1-2) = \hat{\mathbf{x}}x+\hat{\mathbf{y}}(x-3).$$

Since **A** is orthogonal to **B**,

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= 0, \\ [\hat{\mathbf{x}}x+\hat{\mathbf{y}}(x-3)]\cdot(\hat{\mathbf{x}}+\hat{\mathbf{y}}) &= 0 \\ x+x-3 &= 0 \\ x &= \frac{3}{2}\,. \end{aligned}$$

Finally,

$$\begin{aligned} \mathbf{A} = \hat{\mathbf{x}}x+\hat{\mathbf{y}}(x-3) &= \hat{\mathbf{x}}\frac{3}{2}+\hat{\mathbf{y}}\left(\frac{3}{2}-3\right) \\ &= \hat{\mathbf{x}}\frac{3}{2}-\hat{\mathbf{y}}\frac{3}{2}\,. \end{aligned}$$

Problem 3.52 Vector field **E** is given by

$$\mathbf{E} = \hat{\mathbf{R}}\, 5R\cos\theta - \hat{\boldsymbol{\theta}}\frac{12}{R}\sin\theta\cos\phi + \hat{\boldsymbol{\phi}}3\sin\phi.$$

Determine the component of **E** tangential to the spherical surface $R = 2$ at point $P(2, 30°, 60°)$.

Solution: At P, **E** is given by

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{R}}\, 5\times 2\cos 30° - \hat{\boldsymbol{\theta}}\frac{12}{2}\sin 30°\cos 60° + \hat{\boldsymbol{\phi}}3\sin 60° \\ &= \hat{\mathbf{R}}\, 8.67 - \hat{\boldsymbol{\theta}}1.5 + \hat{\boldsymbol{\phi}}2.6.\end{aligned}$$

The $\hat{\mathbf{R}}$ component is normal to the spherical surface while the other two are tangential. Hence,

$$\mathbf{E}_t = -\hat{\boldsymbol{\theta}}1.5 + \hat{\boldsymbol{\phi}}2.6.$$

Problem 3.53 Transform the vector

$$\mathbf{A} = \hat{\mathbf{R}}\sin^2\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos^2\phi - \hat{\boldsymbol{\phi}}\sin\phi$$

into cylindrical coordinates and then evaluate it at $P(2, \pi/2, \pi/2)$.

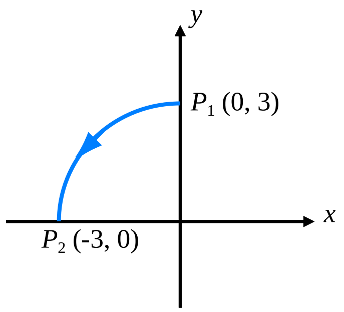
Solution: From Table 3-2,

$$\begin{aligned}\mathbf{A} &= (\hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\sin^2\theta\cos\phi + (\hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta)\cos^2\phi - \hat{\boldsymbol{\phi}}\sin\phi \\ &= \hat{\mathbf{r}}(\sin^3\theta\cos\phi + \cos\theta\cos^2\phi) - \hat{\boldsymbol{\phi}}\sin\phi + \hat{\mathbf{z}}(\cos\theta\sin^2\theta\cos\phi - \sin\theta\cos^2\phi)\end{aligned}$$

At $P(2, \pi/2, \pi/2)$,

$$\mathbf{A} = -\hat{\boldsymbol{\phi}}.$$

Problem 3.54 Evaluate the line integral of $\mathbf{E} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$ along the segment P_1 to P_2 of the circular path shown in the figure.



Solution: We need to calculate:

$$\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell}.$$

Since the path is along the perimeter of a circle, it is best to use cylindrical coordinates, which requires expressing both **E** and $d\boldsymbol{\ell}$ in cylindrical coordinates. Using Table 3-2,

$$\begin{aligned}\mathbf{E} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y &= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi) r\cos\phi - (\hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi) r\sin\phi \\ &= \hat{\mathbf{r}}\, r(\cos^2\phi - \sin^2\phi) - \hat{\boldsymbol{\phi}} 2r\sin\phi\cos\phi\end{aligned}$$

The designated path is along the ϕ-direction at a constant $r = 3$. From Table 3-1, the applicable component of $d\boldsymbol{\ell}$ is:

$$d\boldsymbol{\ell} = \hat{\boldsymbol{\phi}}\, r\, d\phi.$$

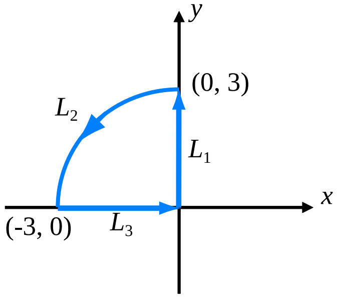
Hence,

$$\begin{aligned}\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} &= \int_{\phi=90^\circ}^{\phi=180^\circ} \left[\hat{\mathbf{r}}\, r(\cos^2\phi - \sin^2\phi) - \hat{\boldsymbol{\phi}}\, 2r\sin\phi\cos\phi\right] \cdot \hat{\boldsymbol{\phi}}\, r\, d\phi \Big|_{r=3} \\ &= \int_{90^\circ}^{180^\circ} -2r^2 \sin\phi\cos\phi\, d\phi \Big|_{r=3} \\ &= -2r^2 \left. \left. \frac{\sin^2\phi}{2} \right|_{\phi=90^\circ}^{180^\circ} \right|_{r=3} = 9.\end{aligned}$$

Problem 3.55 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$ by evaluating:

(a) $\oint_C \mathbf{B} \cdot d\boldsymbol{\ell}$ over the path comprising a quarter section of a circle, as shown in the figure, and

(b) $\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s}$ over the surface of the quarter section.



Solution:

(a)

$$\oint_C \mathbf{B}\cdot d\boldsymbol{\ell} = \int_{L_1} \mathbf{B}\cdot d\boldsymbol{\ell} + \int_{L_2} \mathbf{B}\cdot d\boldsymbol{\ell} + \int_{L_3} \mathbf{B}\cdot d\boldsymbol{\ell}$$

Given the shape of the path, it is best to use cylindrical coordinates. **B** is already expressed in cylindrical coordinates, and we need to choose $d\boldsymbol{\ell}$ in cylindrical coordinates:

$$d\boldsymbol{\ell} = \hat{\mathbf{r}}\ dr + \hat{\boldsymbol{\phi}} r\ d\phi + \hat{\mathbf{z}}\ dz.$$

Along path L_1, $d\phi = 0$ and $dz = 0$. Hence, $d\boldsymbol{\ell} = \hat{\mathbf{r}}\ dr$ and

$$\begin{aligned}\int_{L_1} \mathbf{B}\cdot d\boldsymbol{\ell} &= \int_{r=0}^{r=3} (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot\hat{\mathbf{r}}\ dr\bigg|_{\phi=90^\circ} \\ &= \int_{r=0}^{3} \cos\phi\ dr\bigg|_{\phi=90^\circ} = \ r\cos\phi\big|_{r=0}^{3}\bigg|_{\phi=90^\circ} = 0.\end{aligned}$$

Along L_2, $dr = dz = 0$. Hence, $d\boldsymbol{\ell} = \hat{\boldsymbol{\phi}} r\ d\phi$ and

$$\begin{aligned}\int_{L_2} \mathbf{B}\cdot d\boldsymbol{\ell} &= \int_{\phi=90^\circ}^{180^\circ} (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot\hat{\boldsymbol{\phi}} r\ d\phi\bigg|_{r=3} \\ &= \ -3\cos\phi\big|_{90^\circ}^{180^\circ} = 3.\end{aligned}$$

Along L_3, $dz = 0$ and $d\phi = 0$. Hence, $d\boldsymbol{\ell} = \hat{\mathbf{r}}\ dr$ and

$$\begin{aligned}\int_{L_3} \mathbf{B}\cdot d\boldsymbol{\ell} &= \int_{r=3}^{0} (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot\hat{\mathbf{r}}\ dr\bigg|_{\phi=180^\circ} \\ &= \int_{r=3}^{0} \cos\phi\ dr\big|_{\phi=180^\circ} = \ -r\big|_{3}^{0} = 3.\end{aligned}$$

Hence,

$$\oint_C \mathbf{B}\cdot d\boldsymbol{\ell} = 0 + 3 + 3 = 6.$$

(b)

$$\begin{aligned}\nabla\times\mathbf{B} &= \hat{\mathbf{z}}\,\frac{1}{r}\left(\frac{\partial}{\partial r}\left(rB_\phi - \frac{\partial B_r}{\partial\phi}\right)\right) \\ &= \hat{\mathbf{z}}\,\frac{1}{r}\left(\frac{\partial}{\partial r}(r\sin\phi) - \frac{\partial}{\partial\phi}(\cos\phi)\right) \\ &= \hat{\mathbf{z}}\,\frac{1}{r}(\sin\phi + \sin\phi) = \hat{\mathbf{z}}\,\frac{2}{r}\sin\phi.\end{aligned}$$

$$\begin{aligned}\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s} &= \int_{r=0}^{3}\int_{\phi=90^\circ}^{180^\circ} \left(\hat{\mathbf{z}}\,\frac{2}{r}\sin\phi\right)\cdot\hat{\mathbf{z}}\,r\ dr\ d\phi \\ &= \ -2r\big|_{r=0}^{3}\cos\phi\bigg|_{\phi=90^\circ}^{180^\circ} = 6.\end{aligned}$$

Hence, Stokes's theorem is verified.

Problem 3.56 Find the Laplacian of the following scalar functions:

(a) $V_1 = 10r^3 \sin 2\phi$

(b) $V_2 = (2/R^2) \cos\theta \sin\phi$

Solution:

(a)

$$\begin{aligned}
\nabla^2 V_1 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_1}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(10r^3 \sin 2\phi)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}(10r^3 \sin 2\phi) + 0 \\
&= \frac{1}{r}\frac{\partial}{\partial r}(30r^3 \sin 2\phi) - \frac{1}{r^2}(10r^3) 4 \sin 2\phi \\
&= 90r \sin 2\phi - 40r \sin 2\phi = 50r \sin 2\phi.
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla^2 V_2 &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V_2}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V_2}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V_2}{\partial \phi^2} \\
&= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\left(\frac{2}{R^2}\cos\theta \sin\phi\right)\right) \\
&\quad + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\left(\frac{2}{R^2}\cos\theta \sin\phi\right)\right) \\
&\quad + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}\left(\frac{2}{R^2}\cos\theta \sin\phi\right) \\
&= \frac{4}{R^4}\cos\theta \sin\phi - \frac{4}{R^4}\cos\theta \sin\phi - \frac{2}{R^4}\frac{\cos\theta}{\sin^2\theta}\sin\phi \\
&= -\frac{2}{R^4}\frac{\cos\theta \sin\phi}{\sin^2\theta}\,.
\end{aligned}$$

Chapter 4: Electrostatics

Lesson #22

Chapter — Section: 4-1 to 4-3

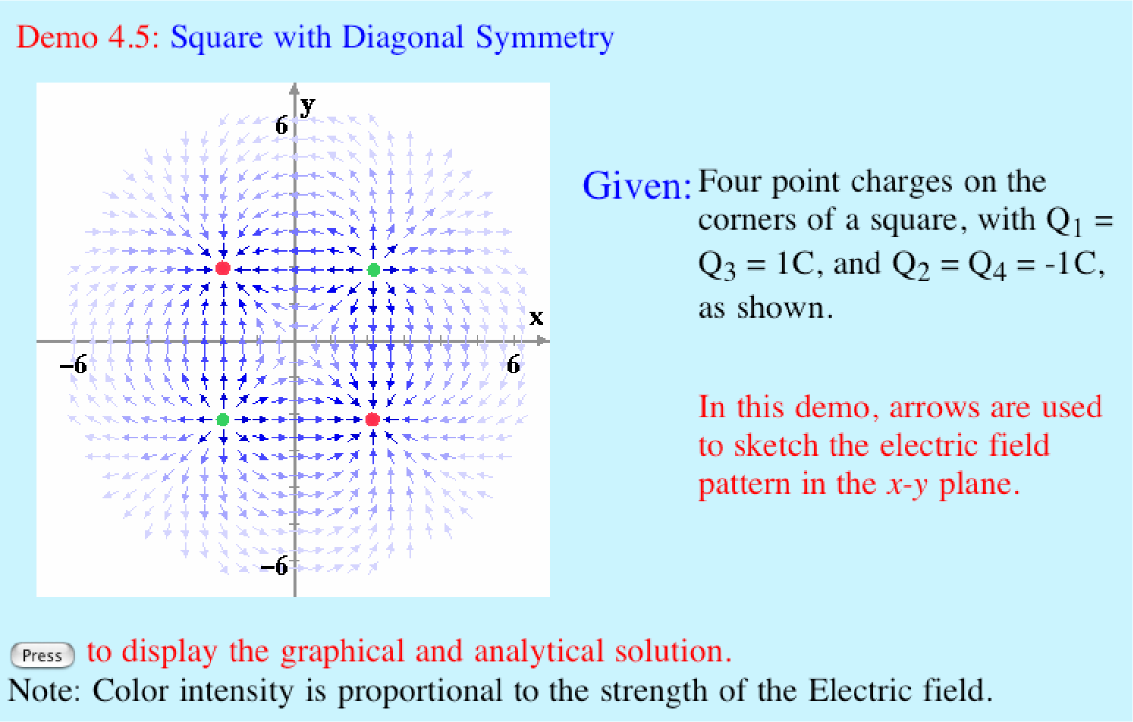
Topics: Charge and current distributions, Coulomb's law

Highlights:

- Maxwell's Equations reduce to *uncoupled* electrostatics and magnetostatics when charges are either fixed in space or move at constant speed.
- Line, surface and volume charge distributions
- Coulomb's law for various charge distributions

Special Illustrations:

- Examples 4-3 and 4-4
- CD-ROM Modules 4.1-4.5
- CD-ROM Demos 4.1-4.8



Demo 4.5: Square with Diagonal Symmetry

Given: Four point charges on the corners of a square, with $Q_1 = Q_3 = 1C$, and $Q_2 = Q_4 = -1C$, as shown.

In this demo, arrows are used to sketch the electric field pattern in the *x*-*y* plane.

Press to display the graphical and analytical solution.

Note: Color intensity is proportional to the strength of the Electric field.

Lesson #23

Chapter — Section: 4-4

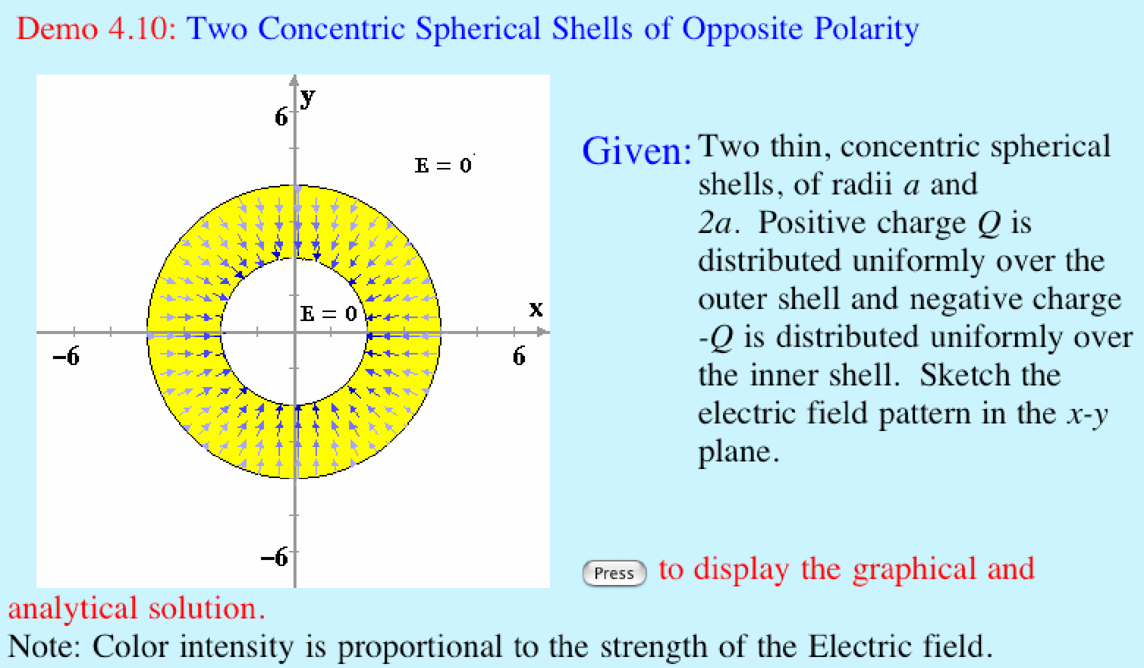
Topics: Gauss's law

Highlights:

- Gauss's law in differential and integral form
- The need for symmetry to apply Gauss's law in practice
- Coulomb's law for various charge distributions

Special Illustrations:

- Example 4-6
- CD-ROM Module 4.6
- CD-ROM Demos 4.9 and 4.10



Demo 4.10: Two Concentric Spherical Shells of Opposite Polarity

Given: Two thin, concentric spherical shells, of radii a and $2a$. Positive charge Q is distributed uniformly over the outer shell and negative charge $-Q$ is distributed uniformly over the inner shell. Sketch the electric field pattern in the x-y plane.

Press to display the graphical and analytical solution.

Note: Color intensity is proportional to the strength of the Electric field.

Lesson #24

Chapter — Section: 4-5

Topics: Electric potential

Highlights:

- Concept of "potential"
- Relation to electric field
- Relation to charges
- Poisson's and Laplace's equations

Special Illustrations:

- Example 4-7

Lesson #25

Chapter — Section: 4-6 and 4-7

Topics: Electrical materials and conductors

Highlights:

- Conductivity ranges for conductors, semiconductors, and insulators
- Ohm's law
- Resistance of a wire
- Joule's law

Special Illustrations:

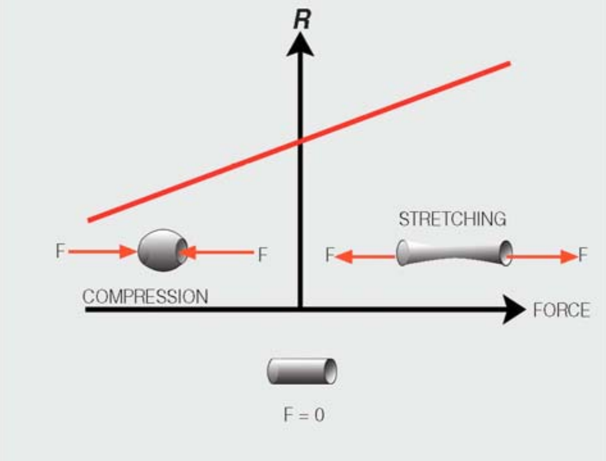
- Example 4-9
- Technology Brief on "Resistive Sensors" (CD-ROM)

Resistive Sensors

An electrical sensor is a device capable of responding to an applied stimulus by generating an electrical signal whose voltage, current, or some other attribute is related to the intensity of the stimulus. The family of possible stimuli encompasses a wide array of physical, chemical, and biological quantities including temperature, pressure, position, distance, motion, velocity, acceleration, concentration (of a gas or liquid), blood flow, etc. The sensing process relies on measuring resistance, capacitance, inductance, induced electromotive force (emf), oscillation frequency or time delay, among others. This Technology Brief covers resistive sensors. Capacitive, inductive, and emf sensors are covered separately (in this and later chapters).

Piezoresistivity

According to Eq. (4.70), the resistance of a cylindrical resistor or wire conductor is given by $R = l/\sigma A$), where l is the cylinder's length, A is its cross-sectional area, and σ is the conductivity of its material. Stretching the wire by an applied external force causes l to increase and A to decrease. Consequently, R increases (A). Conversely, compressing the wire causes R to decrease. The Greek word *piezein* means to press, from which the term piezoresistivity is derived. This should not be confused with piezoelectricity, which is an emf effect (see EMF Sensors).



A1. **Piezoresistance varies with applied force**

Lesson #26

Chapter — Section: 4-8, 4-9

Topics: Dielectrics, boundary conditions

Highlights:

- Relative permittivity and dielectric strength
- Electrostatic boundary conditions for various dielectric and conductor combinations

Special Illustrations:

- Example 4-10

Lesson #27

Chapter — Section: 4-10

Topics: Capacitance

Highlights:

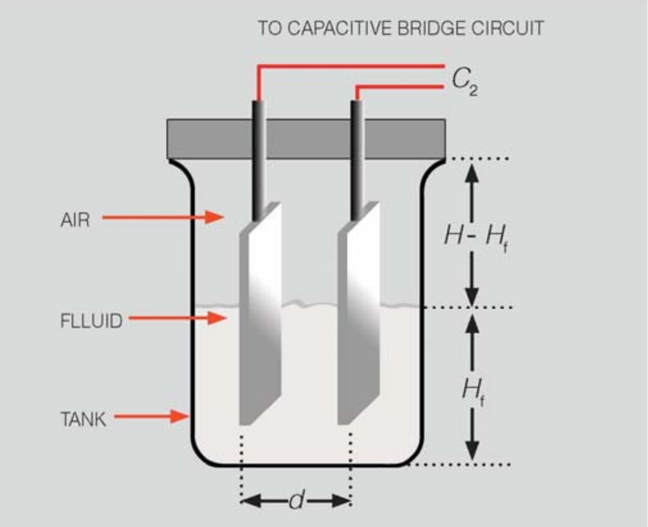
- Capacitor as “charge accumulator”
- General expression for *C*
- Capacitance of parallel-plate and coaxial capacitors
- Joule’s law

Special Illustrations:

- Examples 4-11 and 4-12
- Technology Brief on “Capacitive Sensors” (CD-ROM)

Capacitive Sensors

To sense is to respond to a stimulus (see Resistive Sensors). A capacitor can function as a sensor if the stimulus changes the capacitor’s geometry—usually the spacing between its conductive elements—or the dielectric properties of the insulating material situated between them. Capacitive sensors are used in a multitude of applications. A few examples follow.



A. Fluid tank

Fluid Gauge

The two metal electrodes in (A), usually rods or plates, form a capacitor whose capacitance is directly proportional to the permittivity of the material between them. If the fluid section is of height H_f and the height of the empty space above it is $(H - H_f)$, then the overall capacitance is equivalent to two capacitors in parallel:

$$C_2 = C_f + C_a = \varepsilon_f \frac{(wH_f)}{d} + \varepsilon_a \frac{(H - H_f)}{d}$$

where w is the electrode plate width, d is the spacing between electrodes, and ε_f and ε_a are the permittivities of the fluid and air, respectively.

Lesson #28

Chapter — Section: 4-11

Topics: Energy

Highlights:

- A charged capacitor is an energy storage device
- Energy density

Special Illustrations:

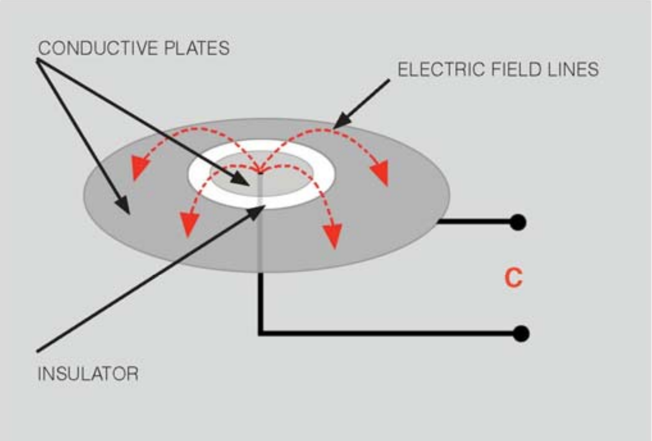
- Technology Brief on "Non-Contact Sensors" (CD-ROM)

Non-Contact Sensors

Precision positioning is a critical ingredient of semiconductor device fabrication, as well as the operation and control of many mechanical systems. Non-contact capacitive sensors are used to sense the position of silicon wafers during the deposition, etching, and cutting processes, without coming in direct contact with the wafers. They are also used to sense and control robot arms in equipment manufacturing and to position hard disc drives, photocopier rollers, printing presses, and other similar systems.

Basic Principle

The concentric plate capacitor (A1) consists of two metal plates, sharing the same plane, but electrically isolated from each other by an insulating material. When connected to a voltage source, charges of opposite polarity will form on the two plates, resulting in the creation of electric-field lines between them. The same principle applies to the adjacent-plates capacitor in (A2). In both cases, the capacitance is determined by the shapes and sizes of the conductive elements and by the permittivity of the dielectric medium containing the electric field lines between them.



A1. **Concentric-plates capacitor**

Lesson #29

Chapter — Section: 4-12

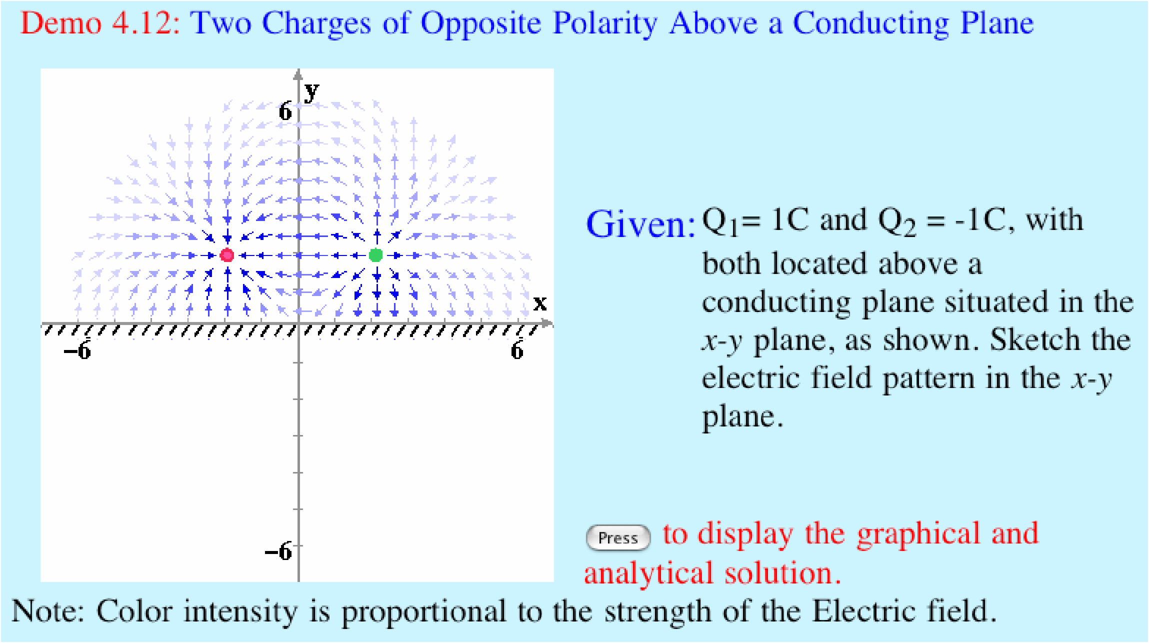
Topics: Image method

Highlights:

- Image method useful for solving problems involving charges next to conducting planes
- Remove conducting plane and replace with mirror images for the charges (with opposite polarity)

Special Illustrations:

- Example 4-13
- CD-ROM Demos 4.11-4.13



Demo 4.12: Two Charges of Opposite Polarity Above a Conducting Plane

Given: Q_1= 1C and Q_2 = -1C, with both located above a conducting plane situated in the *x-y* plane, as shown. Sketch the electric field pattern in the *x-y* plane.

Press to display the graphical and analytical solution.

Note: Color intensity is proportional to the strength of the Electric field.

Chapter 4

Sections 4-2: Charge and Current Distributions

Problem 4.1 A cube 2 m on a side is located in the first octant in a Cartesian coordinate system, with one of its corners at the origin. Find the total charge contained in the cube if the charge density is given by $\rho_v = xy^2e^{-2z}$ (mC/m^3).

Solution: For the cube shown in Fig. P4.1, application of Eq. (4.5) gives

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} xy^2e^{-2z} \, dx \, dy \, dz$$

$$= \left(\frac{-1}{12} x^2y^3e^{-2z} \right) \Bigg|_{x=0}^{2} \Bigg|_{y=0}^{2} \Bigg|_{z=0}^{2} = \frac{8}{3}(1 - e^{-4}) = 2.62 \text{ mC}.$$

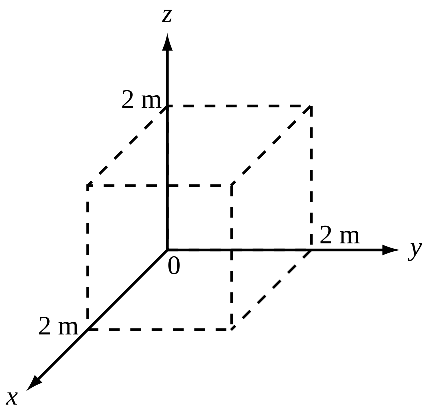


Figure P4.1: Cube of Problem 4.1.

Problem 4.2 Find the total charge contained in a cylindrical volume defined by $r \leq 2$ m and $0 \leq z \leq 3$ m if $\rho_v = 20rz$ (mC/m^3).

Solution: For the cylinder shown in Fig. P4.2, application of Eq. (4.5) gives

$$Q = \int_{z=0}^{3} \int_{\phi=0}^{2\pi} \int_{r=0}^{2} 20rz \, r \, dr \, d\phi \, dz$$

$$= \left(\frac{10}{3} r^3 \phi z^2 \right) \Bigg|_{r=0}^{2} \Bigg|_{\phi=0}^{2\pi} \Bigg|_{z=0}^{3} = 480\pi \text{ (mC)} = 1.5 \text{ C}.$$

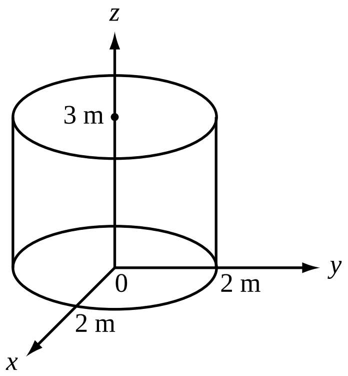


Figure P4.2: Cylinder of Problem 4.2.

Problem 4.3 Find the total charge contained in a cone defined by $R \leq 2$ m and $0 \leq \theta \leq \pi/4$, given that $\rho_v = 10R^2 \cos^2\theta$ (mC/m^3).

Solution: For the cone of Fig. P4.3, application of Eq. (4.5) gives

$$
\begin{aligned}
Q &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \int_{R=0}^{2} 10R^2 \cos^2\theta \, R^2 \sin\theta \, dR \, d\theta \, d\phi \\
&= \left(\frac{-2}{3} R^5 \phi \cos^3\theta \right) \Bigg|_{R=0}^{2} \Bigg|_{\theta=0}^{\pi/4} \Bigg|_{\phi=0}^{2\pi} \\
&= \frac{128\pi}{3} \left(1 - \left(\frac{\sqrt{2}}{2} \right)^3 \right) = 86.65 \text{ (mC)}.
\end{aligned}
$$

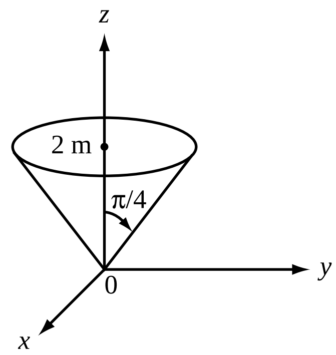


Figure P4.3: Cone of Problem 4.3.

Problem 4.4 If the line charge density is given by $\rho_l = 24y^2$ (mC/m), find the total charge distributed on the y-axis from $y = -5$ to $y = 5$.

Solution:

$$Q = \int_{-5}^{5} \rho_l \, dy = \int_{-5}^{5} 24y^2 \, dy = \left. \frac{24y^3}{3} \right|_{-5}^{5} = 2000 \text{ mC} = 2 \text{ C}.$$

Problem 4.5 Find the total charge on a circular disk defined by $r \le a$ and $z = 0$ if:

(a) $\rho_s = \rho_{s0} \cos\phi$ (C/m^2),

(b) $\rho_s = \rho_{s0} \sin^2\phi$ (C/m^2),

(c) $\rho_s = \rho_{s0} e^{-r}$ (C/m^2),

(d) $\rho_s = \rho_{s0} e^{-r} \sin^2\phi$ (C/m^2),

where ρ_{s0} is a constant.

Solution:

(a)

$$Q = \int \rho_s \, ds = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \cos\phi \; r \, dr \, d\phi = \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \left. \sin\phi \right|_0^{2\pi} = 0.$$

(b)

$$\begin{aligned} Q &= \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \sin^2\phi \; r \, dr \, d\phi = \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \int_0^{2\pi} \left(\frac{1 - \cos 2\phi}{2} \right) d\phi \\ &= \frac{\rho_{s0} a^2}{4} \left. \left(\phi - \frac{\sin 2\phi}{2} \right) \right|_0^{2\pi} = \frac{\pi a^2}{2} \rho_{s0}. \end{aligned}$$

(c)

$$\begin{aligned}Q &= \int_{r=0}^{a}\int_{\phi=0}^{2\pi}\rho_{s0}e^{-r}r\,dr\,d\phi = 2\pi\rho_{s0}\int_{0}^{a}re^{-r}\,dr\\ &= 2\pi\rho_{s0}\left[-re^{-r}-e^{-r}\right]_{0}^{a}\\ &= 2\pi\rho_{s0}[1-e^{-a}(1+a)].\end{aligned}$$

(d)

$$\begin{aligned}Q &= \int_{r=0}^{a}\int_{\phi=0}^{2\pi}\rho_{s0}e^{-r}\sin^2\phi\ r\,dr\,d\phi\\ &= \rho_{s0}\int_{r=0}^{a}re^{-r}\,dr\int_{\phi=0}^{2\pi}\sin^2\phi\,d\phi\\ &= \rho_{s0}[1-e^{-a}(1+a)]\cdot\pi = \pi\rho_{s0}[1-e^{-a}(1+a)].\end{aligned}$$

Problem 4.6 If $\mathbf{J} = \hat{\mathbf{y}}4xz$ (A/m^2), find the current I flowing through a square with corners at $(0,0,0)$, $(2,0,0)$, $(2,0,2)$, and $(0,0,2)$.

Solution: Using Eq. (4.12), the net current flowing through the square shown in Fig. P4.6 is

$$I = \int_{S}\mathbf{J}\cdot d\mathbf{s} = \int_{x=0}^{2}\int_{z=0}^{2}(\hat{\mathbf{y}}4xz)\Big|_{y=0}\cdot(\hat{\mathbf{y}}\,dx\,dz) = \left(x^2z^2\right)\Big|_{x=0}^{2}\Big|_{z=0}^{2} = 16\text{ A}.$$

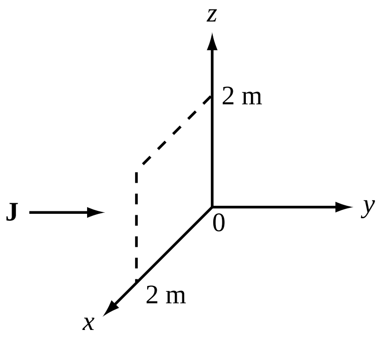


Figure P4.6: Square surface.

Problem 4.7 If $\mathbf{J} = \hat{\mathbf{R}}5/R$ (A/m^2), find I through the surface $R = 5$ m.

Solution: Using Eq. (4.12), we have

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(\hat{\mathbf{R}} \frac{5}{R} \right) \cdot (\hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi)$$
$$= -5R\phi\cos\theta \Big|_{R=5} \Big|_{\theta=0}^{\pi} \Big|_{\phi=0}^{2\pi} = 100\pi = 314.2 \quad \text{(A)}.$$

Problem 4.8 An electron beam shaped like a circular cylinder of radius r_0 carries a charge density given by

$$\rho_v = \left(\frac{-\rho_0}{1+r^2} \right) \quad \text{(C/m}^3\text{)},$$

where ρ_0 is a positive constant and the beam's axis is coincident with the z-axis.

(a) Determine the total charge contained in length L of the beam.

(b) If the electrons are moving in the $+z$-direction with uniform speed u, determine the magnitude and direction of the current crossing the z-plane.

Solution:

(a)

$$Q = \int_{r=0}^{r_0} \int_{z=0}^{L} \rho_v \, d\mathcal{V} = \int_{r=0}^{r_0} \int_{z=0}^{L} \left(\frac{-\rho_0}{1+r^2} \right) 2\pi r \, dr \, dz$$
$$= -2\pi\rho_0 L \int_0^{r_0} \frac{r}{1+r^2} \, dr = -\pi\rho_0 L \ln(1+r_0^2).$$

(b)

$$\mathbf{J} = \rho_v \mathbf{u} = -\hat{\mathbf{z}} \frac{u\rho_0}{1+r^2} \quad \text{(A/m}^2\text{)},$$
$$I = \int \mathbf{J} \cdot d\mathbf{s}$$
$$= \int_{r=0}^{r_0} \int_{\phi=0}^{2\pi} \left(-\hat{\mathbf{z}} \frac{u\rho_0}{1+r^2} \right) \cdot \hat{\mathbf{z}} r \, dr \, d\phi$$
$$= -2\pi u\rho_0 \int_0^{r_0} \frac{r}{1+r^2} \, dr = -\pi u\rho_0 \ln(1+r_0^2) \quad \text{(A)}.$$

Current direction is along $-\hat{\mathbf{z}}$.

Section 4-3: Coulomb's Law

Problem 4.9 A square with sides 2 m each has a charge of 40 μC at each of its four corners. Determine the electric field at a point 5 m above the center of the square.

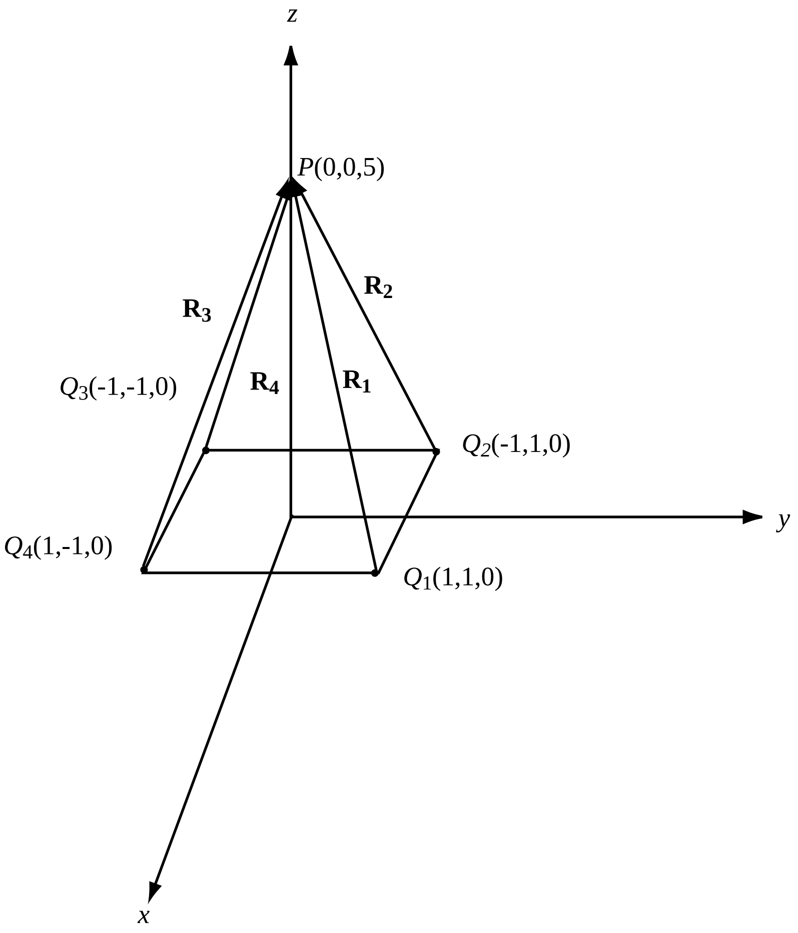


Figure P4.9: Square with charges at the corners.

Solution: The distance $|R|$ between any of the charges and point P is

$|R| = \sqrt{1^2+1^2+5^2} = \sqrt{27}.$

$$\begin{aligned}
\mathbf{E} &= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\mathbf{R}_1}{|\mathbf{R}|^3}+\frac{\mathbf{R}_2}{|\mathbf{R}|^3}+\frac{\mathbf{R}_3}{|\mathbf{R}|^3}+\frac{\mathbf{R}_4}{|\mathbf{R}|^3}\right]\\
&= \frac{Q}{4\pi\varepsilon_0}\left[\frac{-\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{-\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}+\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}}5}{(27)^{3/2}}\right]\\
&= \hat{\mathbf{z}}\,\frac{5Q}{(27)^{3/2}\pi\varepsilon_0} = \hat{\mathbf{z}}\,\frac{5\times 40\ \mu\text{C}}{(27)^{3/2}\pi\varepsilon_0} = \frac{1.42}{\pi\varepsilon_0}\times 10^{-6}\ \text{(V/m)} = \hat{\mathbf{z}}\,51.2\ \text{(kV/m)}.
\end{aligned}$$

Problem 4.10 Three point charges, each with $q = 3$ nC, are located at the corners of a triangle in the x–y plane, with one corner at the origin, another at $(2\text{ cm}, 0, 0)$, and the third at $(0, 2\text{ cm}, 0)$. Find the force acting on the charge located at the origin.

Solution: Use Eq. (4.19) to determine the electric field at the origin due to the other two point charges [Fig. P4.10]:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\left[\frac{3\text{ nC }(-\hat{\mathbf{x}}\,0.02)}{(0.02)^3}\right] + \frac{3\text{ nC }(-\hat{\mathbf{y}}\,0.02)}{(0.02)^3} = -67.4(\hat{\mathbf{x}}+\hat{\mathbf{y}})\text{ (kV/m) at }\mathbf{R}=0.$$

Employ Eq. (4.14) to find the force $\mathbf{F} = q\mathbf{E} = -202.2(\hat{\mathbf{x}}+\hat{\mathbf{y}})$ (μN).

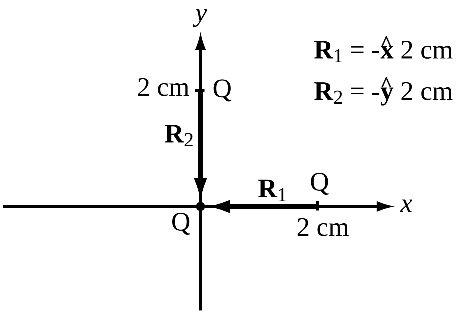


Figure P4.10: Locations of charges in Problem 4.10.

Problem 4.11 Charge $q_1 = 6$ μC is located at $(1\text{ cm}, 1\text{ cm}, 0)$ and charge q_2 is located at $(0, 0, 4\text{ cm})$. What should q_2 be so that $\mathbf{E}$ at $(0, 2\text{ cm}, 0)$ has no y-component?

Solution: For the configuration of Fig. P4.11, use of Eq. (4.19) gives

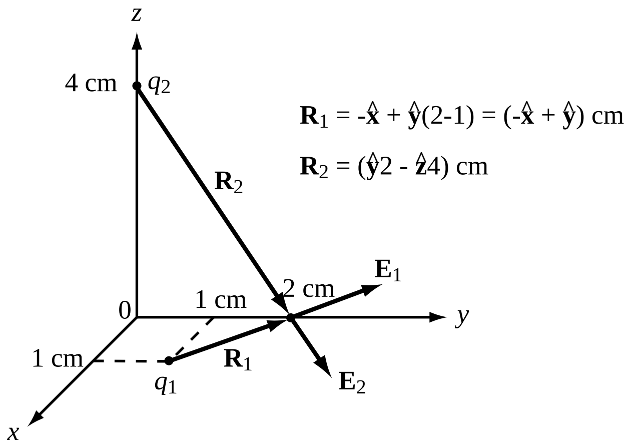


Figure P4.11: Locations of charges in Problem 4.11.

$$\begin{aligned}\mathbf{E}(\mathbf{R}=\hat{\mathbf{y}}2\text{cm}) &= \frac{1}{4\pi\varepsilon}\left[\frac{6\mu\text{C}(-\hat{\mathbf{x}}+\hat{\mathbf{y}})\times 10^{-2}}{(2\times 10^{-2})^{3/2}}+\frac{q_2(\hat{\mathbf{y}}2-\hat{\mathbf{z}}4)\times 10^{-2}}{(20\times 10^{-2})^{3/2}}\right]\\ &= \frac{1}{4\pi\varepsilon}[-\hat{\mathbf{x}}21.21\times 10^{-6}+\hat{\mathbf{y}}(21.21\times 10^{-6}+0.224q_2)\\ &\quad -\hat{\mathbf{z}}0.447q_2] \quad \text{(V/m)}.\end{aligned}$$

If $E_y = 0$, then $q_2 = -21.21\times 10^{-6}/0.224 \approx -94.69$ (μC).

Problem 4.12 A line of charge with uniform density $\rho_l = 8$ (μC/m) exists in air along the z-axis between $z = 0$ and $z = 5$ cm. Find **E** at (0,10 cm,0).

Solution: Use of Eq. (4.21c) for the line of charge shown in Fig. P4.12 gives

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\int_{l'}\hat{\mathbf{R}}'\frac{\rho_l\,dl'}{R'^2},\\ R' &= \hat{\mathbf{y}}0.1-\hat{\mathbf{z}}z\\ &= \frac{1}{4\pi\varepsilon_0}\int_{z=0}^{0.05}(8\times 10^{-6})\frac{(\hat{\mathbf{y}}0.1-\hat{\mathbf{z}}z)}{[(0.1)^2+z^2]^{3/2}}\,dz\\ &= \frac{8\times 10^{-6}}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{y}}10z+\hat{\mathbf{z}}}{\sqrt{(0.1)^2+z^2}}\right]\Bigg|_{z=0}^{0.05}\\ &= 71.86\times 10^3\,[\hat{\mathbf{y}}4.47-\hat{\mathbf{z}}1.06] = \hat{\mathbf{y}}321.4\times 10^3-\hat{\mathbf{z}}76.2\times 10^3 \quad \text{(V/m)}.\end{aligned}$$

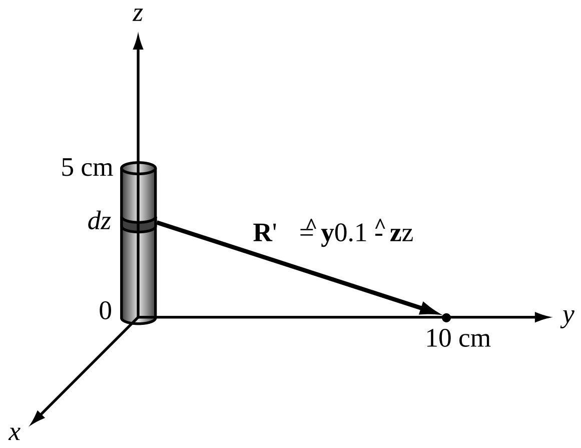


Figure P4.12: Line charge.

Problem 4.13 Electric charge is distributed along an arc located in the x–y plane and defined by $r = 2$ cm and $0 \leq \phi \leq \pi/4$. If $\rho_l = 5$ (μC/m), find $\mathbf{E}$ at $(0,0,z)$ and then evaluate it at (a) the origin, (b) $z = 5$ cm, and (c) $z = -5$ cm.

Solution: For the arc of charge shown in Fig. P4.13, $dl = r\,d\phi = 0.02\,d\phi$, and $\mathbf{R}' = -\hat{\mathbf{x}}0.02\cos\phi - \hat{\mathbf{y}}0.02\sin\phi + \hat{\mathbf{z}}z$. Use of Eq. (4.21c) gives

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\int_{l'} \hat{\mathbf{R}}'\,\frac{\rho_l\,dl'}{R'^2} \\
&= \frac{1}{4\pi\varepsilon_0}\int_{\phi=0}^{\pi/4} \rho_l \frac{(-\hat{\mathbf{x}}0.02\cos\phi - \hat{\mathbf{y}}0.02\sin\phi + \hat{\mathbf{z}}z)}{((0.02)^2+z^2)^{3/2}}\,0.02\,d\phi \\
&= \frac{898.8}{\left((0.02)^2+z^2\right)^{3/2}}\left[-\hat{\mathbf{x}}0.014 - \hat{\mathbf{y}}0.006 + \hat{\mathbf{z}}0.78z\right] \quad \text{(V/m)}.
\end{aligned}
$$

(a) At $z = 0$, $\mathbf{E} = -\hat{\mathbf{x}}\,1.6 - \hat{\mathbf{y}}\,0.66$ (MV/m).
(b) At $z = 5$ cm, $\mathbf{E} = -\hat{\mathbf{x}}\,81.4 - \hat{\mathbf{y}}\,33.7 + \hat{\mathbf{z}}\,226$ (kV/m).
(c) At $z = -5$ cm, $\mathbf{E} = -\hat{\mathbf{x}}\,81.4 - \hat{\mathbf{y}}\,33.7 - \hat{\mathbf{z}}\,226$ (kV/m).

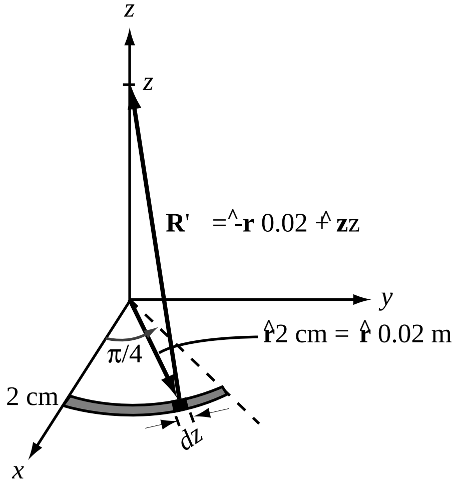


Figure P4.13: Line charge along an arc.

Problem 4.14 A line of charge with uniform density ρ_l extends between $z = -L/2$ and $z = L/2$ along the z-axis. Apply Coulomb's law to obtain an expression for the electric field at any point $P(r,\phi,0)$ on the x–y plane. Show that your result reduces to the expression given by Eq. (4.33) as the length L is extended to infinity.

Solution: Consider an element of charge of height dz at height z. Call it element 1. The electric field at P due to this element is $d\mathbf{E}_1$. Similarly, an element at $-z$ produces $d\mathbf{E}_2$. These two electric fields have equal z-components, but in opposite directions, and hence they will cancel. Their components along $\hat{\mathbf{r}}$ will add. Thus, the net field due to both elements is

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{r}}\frac{2\rho_l \cos\theta \, dz}{4\pi\varepsilon_0 R^2} = \frac{\hat{\mathbf{r}}\rho_l \cos\theta \, dz}{2\pi\varepsilon_0 R^2} \, .$$

where the $\cos\theta$ factor provides the components of $d\mathbf{E}_1$ and $d\mathbf{E}_2$ along $\hat{\mathbf{r}}$.

Our integration variable is z, but it will be easier to integrate over the variable θ from $\theta = 0$ to

$$\theta_0 = \sin^{-1}\frac{L/2}{\sqrt{r^2 + (L/2)^2}} \, .$$

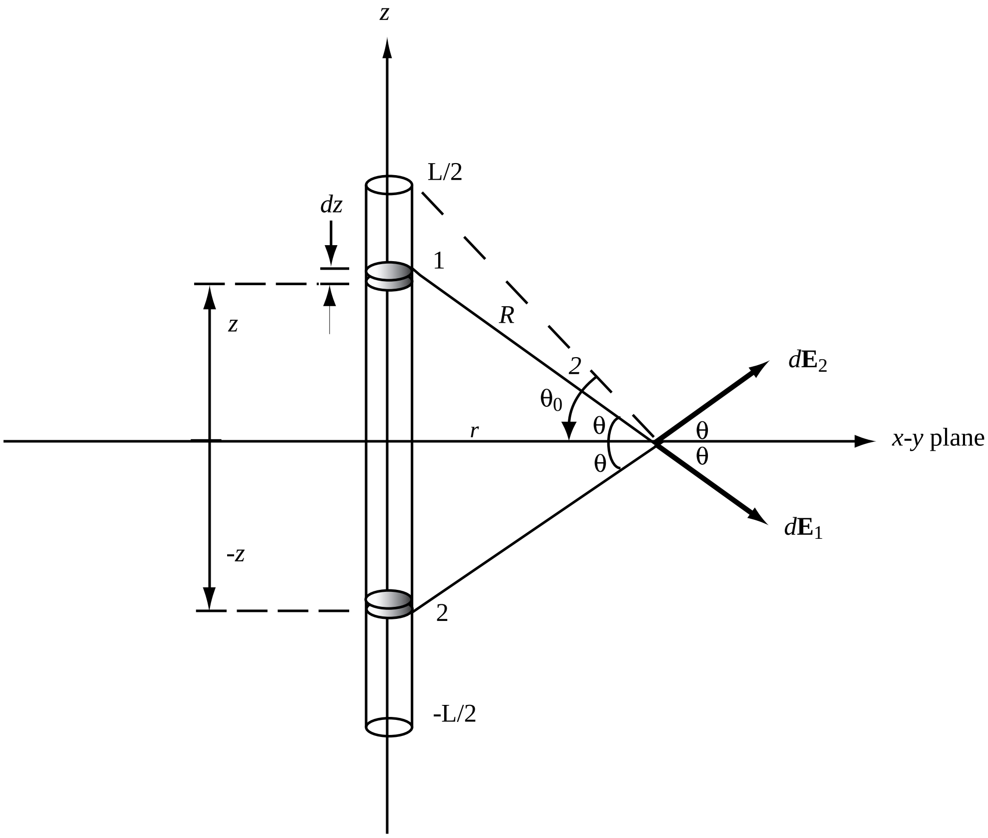


Figure P4.14: Line charge of length L.

Hence, with $R = r/\cos\theta$, and $z = r\tan\theta$ and $dz = r\sec^2\theta\,d\theta$, we have

$$\begin{aligned}\mathbf{E} = \int_{z=0}^{L/2} d\mathbf{E} = \int_{\theta=0}^{\theta_0} d\mathbf{E} &= \int_0^{\theta_0} \hat{\mathbf{r}}\,\frac{\rho_l}{2\pi\varepsilon_0}\,\frac{\cos^3\theta}{r^2}\,r\sec^2\theta\,d\theta \\ &= \hat{\mathbf{r}}\,\frac{\rho_l}{2\pi\varepsilon_0 r}\int_0^{\theta_0}\cos\theta\,d\theta \\ &= \hat{\mathbf{r}}\,\frac{\rho_l}{2\pi\varepsilon_0 r}\sin\theta_0 = \hat{\mathbf{r}}\,\frac{\rho_l}{2\pi\varepsilon_0 r}\,\frac{L/2}{\sqrt{r^2+(L/2)^2}}\,.\end{aligned}$$

For $L \gg r$,

$$\frac{L/2}{\sqrt{r^2+(L/2)^2}} \approx 1,$$

and

$$\mathbf{E} = \hat{\mathbf{r}}\,\frac{\rho_l}{2\pi\varepsilon_0 r} \quad \text{(infinite line of charge)}.$$

Problem 4.15 Repeat Example 4-5 for the circular disk of charge of radius a, but in the present case assume the surface charge density to vary with r as

$$\rho_s = \rho_{s0} r^2 \quad (\mathrm{C/m^2}),$$

where ρ_{s0} is a constant.

Solution: We start with the expression for $d\mathbf{E}$ given in Example 4-5 but we replace ρ_s with $\rho_{s0} r^2$:

$$\begin{aligned}
d\mathbf{E} &= \hat{\mathbf{z}}\,\frac{h}{4\pi\varepsilon_0 (r^2+h^2)^{3/2}}\,(2\pi\rho_{s0} r^3\,dr),\\
\mathbf{E} &= \hat{\mathbf{z}}\,\frac{\rho_{s0} h}{2\varepsilon_0}\int_0^a \frac{r^3\,dr}{(r^2+h^2)^{3/2}}\,.
\end{aligned}$$

To perform the integration, we use

$$\begin{aligned}
R^2 &= r^2 + h^2,\\
2R\,dR &= 2r\,dr,\\
\mathbf{E} &= \hat{\mathbf{z}}\,\frac{\rho_{s0} h}{2\varepsilon_0}\int_h^{(a^2+h^2)^{1/2}} \frac{(R^2-h^2)\,dR}{R^2}\\
&= \hat{\mathbf{z}}\,\frac{\rho_{s0} h}{2\varepsilon_0}\left[\int_h^{(a^2+h^2)^{1/2}} dR - \int_h^{(a^2+h^2)^{1/2}} \frac{h^2}{R^2}\,dR\right]\\
&= \hat{\mathbf{z}}\,\frac{\rho_{s0} h}{2\varepsilon_0}\left[\sqrt{a^2+h^2} + \frac{h^2}{\sqrt{a^2+h^2}} - 2h\right].
\end{aligned}$$

Problem 4.16 Multiple charges at different locations are said to be in equilibrium if the force acting on any one of them is identical in magnitude and direction to the force acting on any of the others. Suppose we have two negative charges, one located at the origin and carrying charge $-9e$, and the other located on the positive x-axis at a distance d from the first one and carrying charge $-36e$. Determine the location, polarity and magnitude of a third charge whose placement would bring the entire system into equilibrium.

Solution: If

$$\mathbf{F}_1 = \text{force on } Q_1,$$

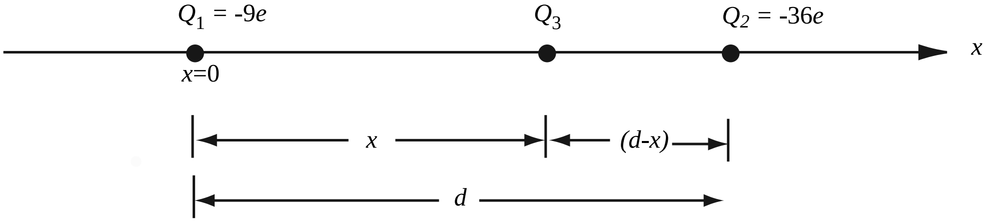


Figure P4.16: Three collinear charges.

$$\mathbf{F}_2 = \text{force on } Q_2,$$
$$\mathbf{F}_3 = \text{force on } Q_3,$$

then equilibrium means that

$$\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}_3.$$

The two original charges are both negative, which mean they would repel each other. The third charge has to be positive and has to lie somewhere between them in order to counteract their repulsion force. The forces acting on charges Q_1, Q_2, and Q_3 are respectively

$$\begin{aligned}
\mathbf{F}_1 &= \frac{\hat{\mathbf{R}}_{21}Q_1Q_2}{4\pi\varepsilon_0R_{21}^2} + \frac{\hat{\mathbf{R}}_{31}Q_1Q_3}{4\pi\varepsilon_0R_{31}^2} = -\hat{\mathbf{x}}\,\frac{324e^2}{4\pi\varepsilon_0 d^2} + \hat{\mathbf{x}}\,\frac{9eQ_3}{4\pi\varepsilon_0 x^2}\,, \\
\mathbf{F}_2 &= \frac{\hat{\mathbf{R}}_{12}Q_1Q_2}{4\pi\varepsilon_0R_{12}^2} + \frac{\hat{\mathbf{R}}_{32}Q_3Q_2}{4\pi\varepsilon_0R_{32}^2} = \hat{\mathbf{x}}\,\frac{324e^2}{4\pi\varepsilon_0 d^2} - \hat{\mathbf{x}}\,\frac{36eQ_3}{4\pi\varepsilon_0 (d-x)^2}\,, \\
\mathbf{F}_3 &= \frac{\hat{\mathbf{R}}_{13}Q_1Q_3}{4\pi\varepsilon_0R_{13}^2} + \frac{\hat{\mathbf{R}}_{23}Q_2Q_3}{4\pi\varepsilon_0R_{23}^2} = -\hat{\mathbf{x}}\,\frac{9eQ_3}{4\pi\varepsilon_0 x^2} + \hat{\mathbf{x}}\,\frac{36eQ_3}{4\pi\varepsilon_0 (d-x)^2}\,.
\end{aligned}$$

Hence, equilibrium requires that

$$-\frac{324e}{d^2} + \frac{9Q_3}{x^2} = \frac{324e}{d^2} - \frac{36Q_3}{(d-x)^2} = -\frac{9Q_3}{x^2} + \frac{36Q_3}{(d-x)^2}\,.$$

Solution of the above equations yields

$$Q_3 = 4e, \qquad x = \frac{d}{3}\,.$$

Section 4-4: Gauss's Law

Problem 4.17 Three infinite lines of charge, all parallel to the z-axis, are located at the three corners of the kite-shaped arrangement shown in Fig. 4-29 (P4.17). If the two right triangles are symmetrical and of equal corresponding sides, show that the electric field is zero at the origin.

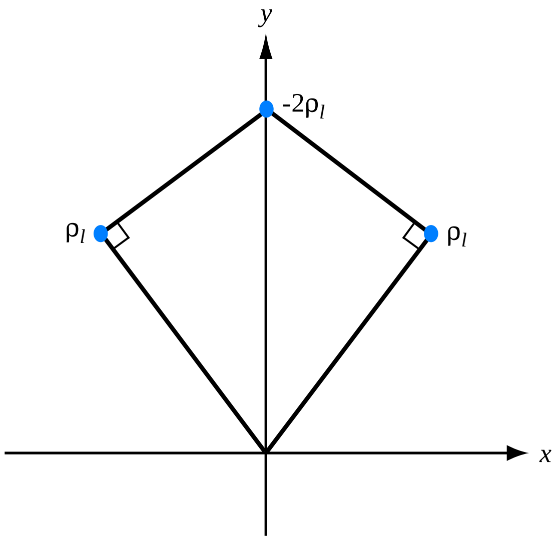


Figure P4.17: Kite-shaped arrangment of line charges for Problem 4.17.

Solution: The field due to an infinite line of charge is given by Eq. (4.33). In the present case, the total **E** at the origin is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3.$$

The components of $\mathbf{E}_1$ and $\mathbf{E}_2$ along $\hat{\mathbf{x}}$ cancel and their components along $-\hat{\mathbf{y}}$ add. Also, $\mathbf{E}_3$ is along $\hat{\mathbf{y}}$ because the line charge on the y-axis is negative. Hence,

$$\mathbf{E} = -\hat{\mathbf{y}}\,\frac{2\rho_l \cos\theta}{2\pi\varepsilon_0 R_1} + \hat{\mathbf{y}}\,\frac{2\rho_l}{2\pi\varepsilon_0 R_2}\,.$$

But $\cos\theta = R_1/R_2$. Hence,

$$\mathbf{E} = -\hat{\mathbf{y}}\,\frac{\rho_l}{\pi\varepsilon_0 R_1}\,\frac{R_1}{R_2} + \hat{\mathbf{y}}\,\frac{\rho_l}{\pi\varepsilon_0 R_2} = 0.$$

Problem 4.18 Three infinite lines of charge, $\rho_{l_1} = 3$ (nC/m), $\rho_{l_2} = -3$ (nC/m), and $\rho_{l_3} = 3$ (nC/m), are all parallel to the z-axis. If they pass through the respective points

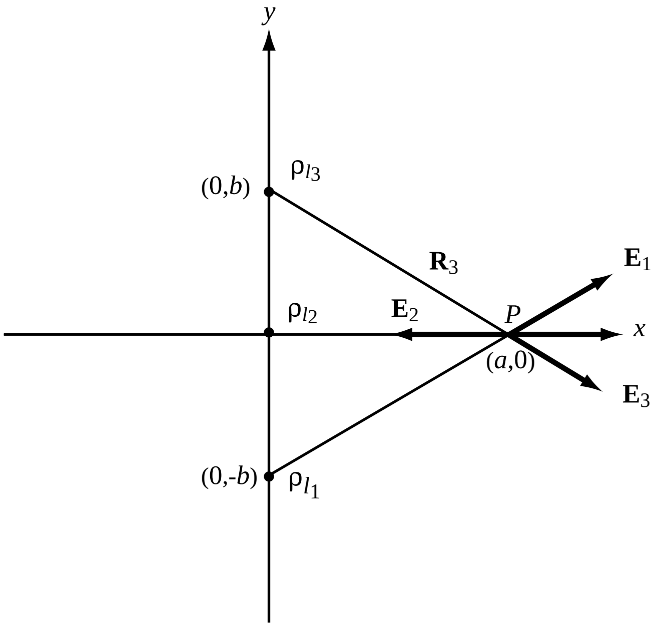


Figure P4.18: Three parallel line charges.

$(0,-b)$, $(0,0)$, and $(0,b)$ in the x–y plane, find the electric field at $(a,0,0)$. Evaluate your result for $a = 2$ cm and $b = 1$ cm.

Solution:

$$\begin{aligned}\rho_{l_1} &= 3 \quad \text{(nC/m)},\\ \rho_{l_2} &= -3 \quad \text{(nC/m)},\\ \rho_{l_3} &= \rho_{l_1},\\ \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3.\end{aligned}$$

Components of line charges 1 and 3 along y cancel and components along x add. Hence, using Eq. (4.33),

$$\mathbf{E} = \hat{\mathbf{x}}\,\frac{2\rho_{l_1}}{2\pi\varepsilon_0 R_1}\cos\theta + \hat{\mathbf{x}}\,\frac{\rho_{l_2}}{2\pi\varepsilon_0 a}\,.$$

with $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$ and $R_1 = \sqrt{a^2+b^2}$,

$$\mathbf{E} = \frac{\hat{\mathbf{x}}\,3}{2\pi\varepsilon_0}\left[\frac{2a}{a^2+b^2} - \frac{1}{a}\right] \times 10^{-9} \quad \text{(V/m)}.$$

For $a = 2$ cm and $b = 1$ cm,

$$\mathbf{E} = \hat{\mathbf{x}}1.62 \quad \text{(kV/m)}.$$

Problem 4.19 A horizontal strip lying in the x–y plane is of width d in the y-direction and infinitely long in the x-direction. If the strip is in air and has a uniform charge distribution ρ_s, use Coulomb's law to obtain an explicit expression for the electric field at a point P located at a distance h above the centerline of the strip. Extend your result to the special case where d is infinite and compare it with Eq. (4.25).

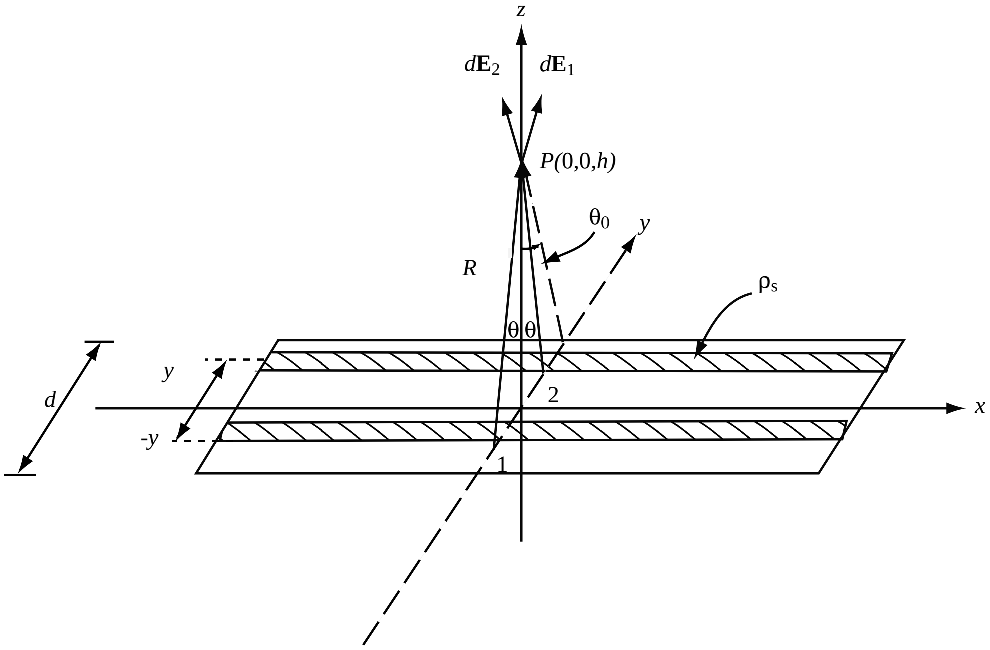


Figure P4.19: Horizontal strip of charge.

Solution: The strip of charge density ρ_s (C/m^2) can be treated as a set of adjacent line charges each of charge $\rho_l = \rho_s\, dy$ and width dy. At point P, the fields of line charge at distance y and line charge at distance $-y$ give contributions that cancel each other along $\hat{\mathbf{y}}$ and add along $\hat{\mathbf{z}}$. For each such pair,

$$d\mathbf{E} = \hat{\mathbf{z}}\,\frac{2\rho_s\, dy \cos\theta}{2\pi\varepsilon_0 R}\,.$$

With $R = h/\cos\theta$, we integrate from $y = 0$ to $d/2$, which corresponds to $\theta = 0$ to $\theta_0 = \sin^{-1}[(d/2)/(h^2 + (d/2)^2)^{1/2}]$. Thus,

$$\mathbf{E} = \int_0^{d/2} d\mathbf{E} = \hat{\mathbf{z}}\,\frac{\rho_s}{\pi\varepsilon_0}\int_0^{d/2}\frac{\cos\theta}{R}\,dy = \hat{\mathbf{z}}\,\frac{\rho_s}{\pi\varepsilon_0}\int_0^{\theta_0}\frac{\cos^2\theta}{h}\cdot\frac{h}{\cos^2\theta}\,d\theta$$

$$= \hat{\mathbf{z}}\,\frac{\rho_s}{\pi\varepsilon_0}\,\theta_0.$$

For an infinitely wide sheet, $\theta_0 = \pi/2$ and $\mathbf{E} = \hat{\mathbf{z}}\,\dfrac{\rho_s}{2\varepsilon_0}$, which is identical with Eq. (4.25).

Problem 4.20 Given the electric flux density

$$\mathbf{D} = \hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y) \quad (\text{C/m}^2),$$

determine

(a) ρ_v by applying Eq. (4.26),

(b) the total charge Q enclosed in a cube 2 m on a side, located in the first octant with three of its sides coincident with the x-, y-, and z-axes and one of its corners at the origin, and

(c) the total charge Q in the cube, obtained by applying Eq. (4.29).

Solution:

(a) By applying Eq. (4.26)

$$\rho_v = \nabla\cdot\mathbf{D} = \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(3x-2y) = 0.$$

(b) Integrate the charge density over the volume as in Eq. (4.27):

$$Q = \int_{\mathcal{V}} \nabla\cdot\mathbf{D}\,d\mathcal{V} = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} 0\,dx\,dy\,dz = 0.$$

(c) Apply Gauss' law to calculate the total charge from Eq. (4.29)

$$Q = \oint \mathbf{D}\cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}},$$

$$F_{\text{front}} = \int_{y=0}^{2}\int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{x=2} \cdot (\hat{\mathbf{x}}\,dz\,dy)$$

$$= \int_{y=0}^{2}\int_{z=0}^{2} 2(x+y)\Big|_{x=2}\,dz\,dy = \left(2z\left(2y + \frac{1}{2}y^2\right)\Big|_{z=0}^{2}\right)\Bigg|_{y=0}^{2} = 24,$$

$$F_{\text{back}} = \int_{y=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\bigg|_{x=0} \cdot (-\hat{\mathbf{x}}\,dz\,dy)$$

$$= -\int_{y=0}^{2}\int_{z=0}^{2} 2(x+y)\bigg|_{x=0} dz\,dy = -\left(zy^2 \bigg|_{z=0}^{2} \right)\bigg|_{y=0}^{2} = -8,$$

$$F_{\text{right}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\bigg|_{y=2} \cdot (\hat{\mathbf{y}}\,dz\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\bigg|_{y=2} dz\,dx = \left(z\left(\frac{3}{2}x^2-4x\right)\bigg|_{z=0}^{2} \right)\bigg|_{x=0}^{2} = -4,$$

$$F_{\text{left}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\bigg|_{y=0} \cdot (-\hat{\mathbf{y}}\,dz\,dx)$$

$$= -\int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\bigg|_{y=0} dz\,dx = -\left(z\left(\frac{3}{2}x^2\right)\bigg|_{z=0}^{2} \right)\bigg|_{x=0}^{2} = -12,$$

$$F_{\text{top}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\bigg|_{z=2} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\bigg|_{z=2} dy\,dx = 0,$$

$$F_{\text{bottom}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\bigg|_{z=0} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\bigg|_{z=0} dy\,dx = 0.$$

Thus $Q = \oint \mathbf{D}\cdot d\mathbf{s} = 24-8-4-12+0+0 = 0$.

Problem 4.21 Repeat Problem 4.20 for $\mathbf{D} = \hat{\mathbf{x}}xy^3z^3$ (C/m^2).

Solution:

(a) From Eq. (4.26), $\rho_{\text{v}} = \nabla\cdot\mathbf{D} = \dfrac{\partial}{\partial x}(xy^3z^3) = y^3z^3$.

(b) Total charge Q is given by Eq. (4.27):

$$Q = \int_{\mathcal{V}} \nabla\cdot\mathbf{D}\,d\mathcal{V} = \int_{z=0}^{2}\int_{y=0}^{2}\int_{x=0}^{2} y^3z^3\,dx\,dy\,dz = \frac{xy^4z^4}{16}\bigg|_{x=0}^{2}\bigg|_{y=0}^{2}\bigg|_{z=0}^{2} = 32 \text{ C}.$$

(c) Using Gauss' law we have

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}.$$

Note that $\mathbf{D} = \hat{\mathbf{x}}D_x$, so only F_{front} and F_{back} (integration over $\hat{\mathbf{z}}$ surfaces) will contribute to the integral.

$$\begin{aligned}
F_{\text{front}} &= \int_{z=0}^{2}\int_{y=0}^{2} (\hat{\mathbf{x}}xy^3z^3)\bigg|_{x=2} \cdot (\hat{\mathbf{x}}\,dy\,dz) \\
&= \int_{z=0}^{2}\int_{y=0}^{2} xy^3z^3\bigg|_{x=2} dy\,dz = \left(2\left(\frac{y^4z^4}{16}\right)\bigg|_{y=0}^{2}\right)\bigg|_{z=0}^{2} = 32\,, \\
F_{\text{back}} &= \int_{z=0}^{2}\int_{y=0}^{2} (\hat{\mathbf{x}}xy^3z^3)\bigg|_{x=0} \cdot (-\hat{\mathbf{x}}\,dy\,dz) = -\int_{z=0}^{2}\int_{y=0}^{2} xy^3z^3\bigg|_{x=0} dy\,dz = 0.
\end{aligned}$$

Thus $Q = \oint \mathbf{D}\cdot d\mathbf{s} = 32+0+0+0+0+0 = 32$ C.

Problem 4.22 Charge Q_1 is uniformly distributed over a thin spherical shell of radius a, and charge Q_2 is uniformly distributed over a second spherical shell of radius b, with $b > a$. Apply Gauss's law to find $\mathbf{E}$ in the regions $R < a$, $a < R < b$, and $R > b$.

Solution: Using symmetry considerations, we know $\mathbf{D} = \hat{\mathbf{R}}D_{\text{R}}$. From Table 3.1, $d\mathbf{s} = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ for an element of a spherical surface. Using Gauss's law in integral form (Eq. (4.29)),

$$\oint_{\text{S}} \mathbf{D}\cdot d\mathbf{s} = Q_{\text{tot}},$$

where Q_{tot} is the total charge enclosed in S. For a spherical surface of radius R,

$$\begin{aligned}
\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (\hat{\mathbf{R}}D_{\text{R}})\cdot(\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi) &= Q_{\text{tot}}, \\
D_{\text{R}}R^2(2\pi)[-\cos\theta]_0^{\pi} &= Q_{\text{tot}}, \\
D_{\text{R}} &= \frac{Q_{\text{tot}}}{4\pi R^2}\,.
\end{aligned}$$

From Eq. (4.15), we know a linear, isotropic material has the constitutive relationship $\mathbf{D} = \varepsilon\mathbf{E}$. Thus, we find $\mathbf{E}$ from $\mathbf{D}$.

(a) In the region $R < a$,

$$Q_{\rm tot} = 0, \qquad \mathbf{E} = \hat{\mathbf{R}}E_{\rm R} = \frac{\hat{\mathbf{R}}Q_{\rm tot}}{4\pi R^2\varepsilon} = 0 \quad \text{(V/m)}.$$

(b) In the region $a < R < b$,

$$Q_{\rm tot} = Q_1, \qquad \mathbf{E} = \hat{\mathbf{R}}E_{\rm R} = \frac{\hat{\mathbf{R}}Q_1}{4\pi R^2\varepsilon} \quad \text{(V/m)}.$$

(c) In the region $R > b$,

$$Q_{\rm tot} = Q_1 + Q_2, \qquad \mathbf{E} = \hat{\mathbf{R}}E_{\rm R} = \frac{\hat{\mathbf{R}}(Q_1 + Q_2)}{4\pi R^2\varepsilon} \quad \text{(V/m)}.$$

Problem 4.23 The electric flux density inside a dielectric sphere of radius a centered at the origin is given by

$$\mathbf{D} = \hat{\mathbf{R}}\rho_0 R \quad \text{(C/m}^2\text{)},$$

where ρ_0 is a constant. Find the total charge inside the sphere.

Solution:

$$\begin{aligned} Q &= \oint_S \mathbf{D}\cdot d\mathbf{s} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \hat{\mathbf{R}}\rho_0 R\cdot\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi\bigg|_{R=a} \\ &= 2\pi\rho_0 a^3\int_0^{\pi}\sin\theta\,d\theta = -2\pi\rho_0 a^3\cos\theta\big|_0^{\pi} = 4\pi\rho_0 a^3 \quad \text{(C)}. \end{aligned}$$

Problem 4.24 In a certain region of space, the charge density is given in cylindrical coordinates by the function:

$$\rho_{\rm v} = 50re^{-r} \quad \text{(C/m}^3\text{)}.$$

Apply Gauss's law to find **D**.

Solution:

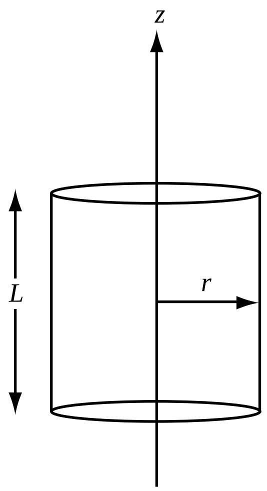


Figure P4.24: Gaussian surface.

Method 1: Integral Form of Gauss's Law

Since ρ_v varies as a function of r only, so will $\mathbf{D}$. Hence, we construct a cylinder of radius r and length L, coincident with the z-axis. Symmetry suggests that $\mathbf{D}$ has the functional form $\mathbf{D} = \hat{\mathbf{r}} D$. Hence,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q,$$

$$\int \hat{\mathbf{r}} D \cdot d\mathbf{s} = D(2\pi r L),$$

$$\begin{aligned} Q &= 2\pi L \int_0^r 50re^{-r} \cdot r\,dr \\ &= 100\pi L[-r^2e^{-r} + 2(1 - e^{-r}(1+r))], \end{aligned}$$

$$\mathbf{D} = \hat{\mathbf{r}} D = \hat{\mathbf{r}} 50 \left[\frac{2}{r}(1 - e^{-r}(1+r)) - re^{-r} \right].$$

Method 2: Differential Method

$$\nabla \cdot \mathbf{D} = \rho_v, \qquad \mathbf{D} = \hat{\mathbf{r}} D_r,$$

with D_r being a function of r.

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) = 50re^{-r},$$

$$\frac{\partial}{\partial r}(rD_r) = 50r^2e^{-r},$$
$$\int_0^r \frac{\partial}{\partial r}(rD_r)\,dr = \int_0^r 50r^2e^{-r}\,dr,$$
$$rD_r = 50[2(1-e^{-r}(1+r)) - r^2e^{-r}],$$
$$\mathbf{D} = \hat{\mathbf{r}}\,rD_r = \hat{\mathbf{r}}\,50\left[\frac{2}{r}(1-e^{-r}(1+r)) - re^{-r}\right].$$

Problem 4.25 An infinitely long cylindrical shell extending between $r = 1$ m and $r = 3$ m contains a uniform charge density ρ_{v0}. Apply Gauss's law to find **D** in all regions.

Solution: For $r < 1$ m, $\mathbf{D} = 0$.
For $1 \le r \le 3$ m,

$$\oint_S \hat{\mathbf{r}}D_r \cdot d\mathbf{s} = Q,$$
$$D_r \cdot 2\pi rL = \rho_{v0} \cdot \pi L(r^2 - 1^2),$$
$$\mathbf{D} = \hat{\mathbf{r}}D_r = \hat{\mathbf{r}}\,\frac{\rho_{v0}\pi L(r^2-1)}{2\pi rL} = \hat{\mathbf{r}}\,\frac{\rho_{v0}(r^2-1)}{2r}, \qquad 1 \le r \le 3 \text{ m}.$$

For $r \ge 3$ m,

$$D_r \cdot 2\pi rL = \rho_{v0}\pi L(3^2 - 1^2) = 8\rho_{v0}\pi L,$$
$$\mathbf{D} = \hat{\mathbf{r}}D_r = \hat{\mathbf{r}}\,\frac{4\rho_{v0}}{r}, \qquad r \ge 3 \text{ m}.$$

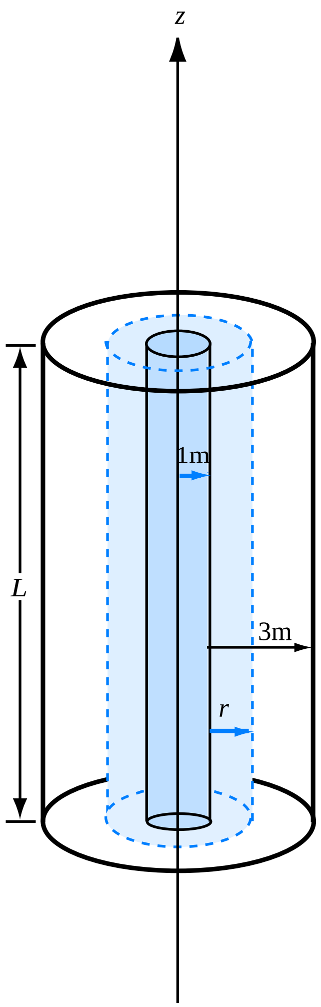


Figure P4.25: Cylindrical shell.

Problem 4.26 If the charge density increases linearly with distance from the origin such that $\rho_v = 0$ at the origin and $\rho_v = 40$ C/m^3 at $R = 2$ m, find the corresponding variation of **D**.

Solution:

$$\rho_v(R) = a + bR,$$
$$\rho_v(0) = a = 0,$$

$$\rho_v(2) = 2b = 40.$$

Hence, $b = 20$.

$$\rho_v(R) = 20R \quad (\text{C/m}^3).$$

Applying Gauss's law to a spherical surface of radius R,

$$\begin{aligned}
\oint_S \mathbf{D} \cdot d\mathbf{s} &= \int_V \rho_v \, d\mathcal{V}, \\
D_R \cdot 4\pi R^2 &= \int_0^R 20R \cdot 4\pi R^2 \, dR = 80\pi \frac{R^4}{4}, \\
D_R &= 5R^2 \quad (\text{C/m}^2), \\
\mathbf{D} &= \hat{\mathbf{R}} D_R = \hat{\mathbf{R}} 5R^2 \quad (\text{C/m}^2).
\end{aligned}$$

Section 4-5: Electric Potential

Problem 4.27 A square in the x–y plane in free space has a point charge of $+Q$ at corner $(a/2, a/2)$ and the same at corner $(a/2, -a/2)$ and a point charge of $-Q$ at each of the other two corners.

(a) Find the electric potential at any point P along the x-axis.

(b) Evaluate V at $x = a/2$.

Solution: $R_1 = R_2$ and $R_3 = R_4$.

$$V = \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{Q}{4\pi\varepsilon_0 R_2} + \frac{-Q}{4\pi\varepsilon_0 R_3} + \frac{-Q}{4\pi\varepsilon_0 R_4} = \frac{Q}{2\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_3}\right)$$

with

$$\begin{aligned}
R_1 &= \sqrt{\left(x - \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}, \\
R_3 &= \sqrt{\left(x + \frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}.
\end{aligned}$$

At $x = a/2$,

$$\begin{aligned}
R_1 &= \frac{a}{2}, \\
R_3 &= \frac{a\sqrt{5}}{2}, \\
V &= \frac{Q}{2\pi\varepsilon_0}\left(\frac{2}{a} - \frac{2}{\sqrt{5}\,a}\right) = \frac{0.55Q}{\pi\varepsilon_0 a}.
\end{aligned}$$

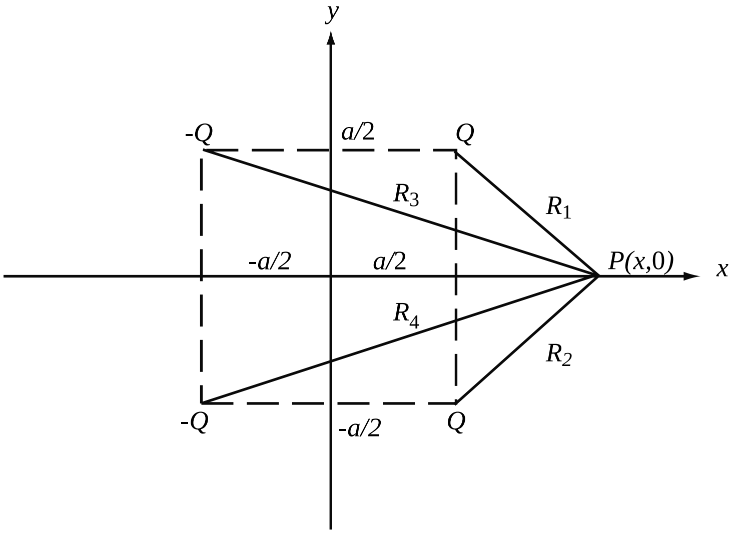


Figure P4.27: Potential due to four point charges.

Problem 4.28 The circular disk of radius a shown in Fig. 4-7 (P4.28) has uniform charge density ρ_s across its surface.

(a) Obtain an expression for the electric potential V at a point $P(0,0,z)$ on the z-axis.

(b) Use your result to find **E** and then evaluate it for $z = h$. Compare your final expression with Eq. (4.24), which was obtained on the basis of Coulomb's law.

Solution:

(a) Consider a ring of charge at a radial distance r. The charge contained in width dr is

$$dq = \rho_s(2\pi r\,dr) = 2\pi\rho_s r\,dr.$$

The potential at P is

$$dV = \frac{dq}{4\pi\varepsilon_0 R} = \frac{2\pi\rho_s r\,dr}{4\pi\varepsilon_0(r^2+z^2)^{1/2}}\,.$$

The potential due to the entire disk is

$$V = \int_0^a dV = \frac{\rho_s}{2\varepsilon_0}\int_0^a \frac{r\,dr}{(r^2+z^2)^{1/2}} = \frac{\rho_s}{2\varepsilon_0}\,(r^2+z^2)^{1/2}\bigg|_0^a = \frac{\rho_s}{2\varepsilon_0}\left[(a^2+z^2)^{1/2}-z\right].$$

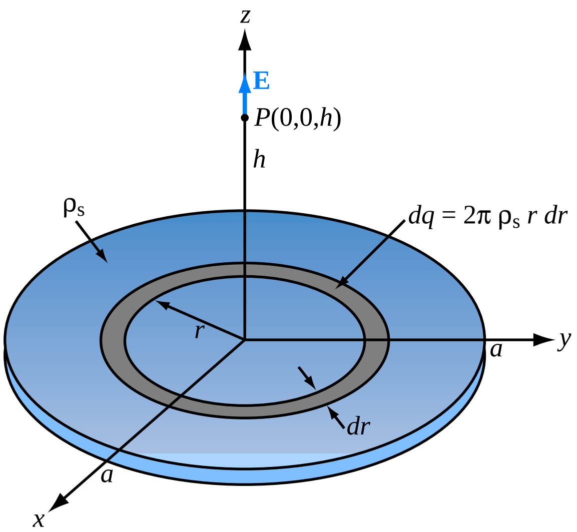


Figure P4.28: Circular disk of charge.

(b)

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{x}}\,\frac{\partial V}{\partial x} - \hat{\mathbf{y}}\,\frac{\partial V}{\partial y} - \hat{\mathbf{z}}\,\frac{\partial V}{\partial z} = \hat{\mathbf{z}}\,\frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right].$$

The expression for **E** reduces to Eq. (4.24) when $z = h$.

Problem 4.29 A circular ring of charge of radius a lies in the x–y plane and is centered at the origin. If the ring is in air and carries a uniform density ρ_l, (a) show that the electrical potential at $(0,0,z)$ is given by $V = \rho_l a/[2\varepsilon_0(a^2+z^2)^{1/2}]$, and (b) find the corresponding electric field **E**.

Solution:

(a) For the ring of charge shown in Fig. P4.29, using Eq. (3.67) in Eq. (4.48c) gives

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon_0}\int_{l'} \frac{\rho_l}{R'}\,dl' = \frac{1}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi} \frac{\rho_l}{\sqrt{a^2+r^2-2ar\cos(\phi'-\phi)+z^2}}\,a\,d\phi'.$$

Point $(0,0,z)$ in Cartesian coordinates corresponds to $(r,\phi,z) = (0,\phi,z)$ in cylindrical coordinates. Hence, for $r = 0$,

$$V(0,0,z) = \frac{1}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi} \frac{\rho_l}{\sqrt{a^2+z^2}}\,a\,d\phi' = \frac{\rho_l a}{2\varepsilon_0\sqrt{a^2+z^2}}.$$

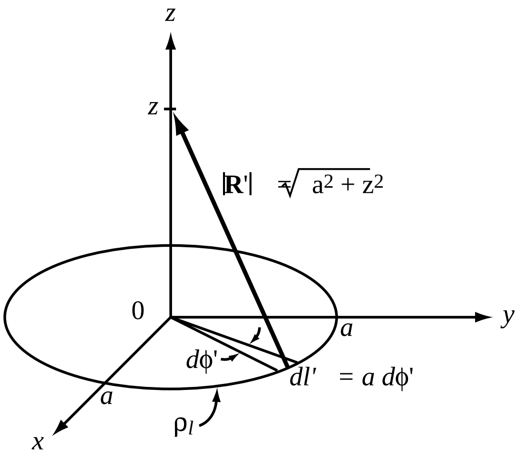


Figure P4.29: Ring of charge.

(b) From Eq. (4.51),

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{z}}\frac{\rho_l a}{2\varepsilon_0}\frac{\partial}{\partial z}\left(a^2+z^2\right)^{-1/2} = \hat{\mathbf{z}}\frac{\rho_l a}{2\varepsilon_0}\frac{z}{(a^2+z^2)^{3/2}} \quad \text{(V/m).}$$

Problem 4.30 Show that the electric potential difference V_{12} between two points in air at radial distances r_1 and r_2 from an infinite line of charge with density ρ_l along the z-axis is $V_{12} = (\rho_l/2\pi\varepsilon_0)\ln(r_2/r_1)$.

Solution: From Eq. (4.33), the electric field due to an infinite line of charge is

$$\mathbf{E} = \hat{\mathbf{r}}E_\mathrm{r} = \hat{\mathbf{r}}\,\frac{\rho_l}{2\pi\varepsilon_0 r}\,.$$

Hence, the potential difference is

$$V_{12} = -\int_{r_2}^{r_1}\mathbf{E}\cdot d\mathbf{l} = -\int_{r_2}^{r_1}\frac{\hat{\mathbf{r}}\rho_l}{2\pi\varepsilon_0 r}\cdot\hat{\mathbf{r}}\,dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r_2}{r_1}\right).$$

Problem 4.31 Find the electric potential V at a location a distance b from the origin in the x–y plane due to a line charge with charge density ρ_l and of length l. The line charge is coincident with the z-axis and extends from $z = -l/2$ to $z = l/2$.

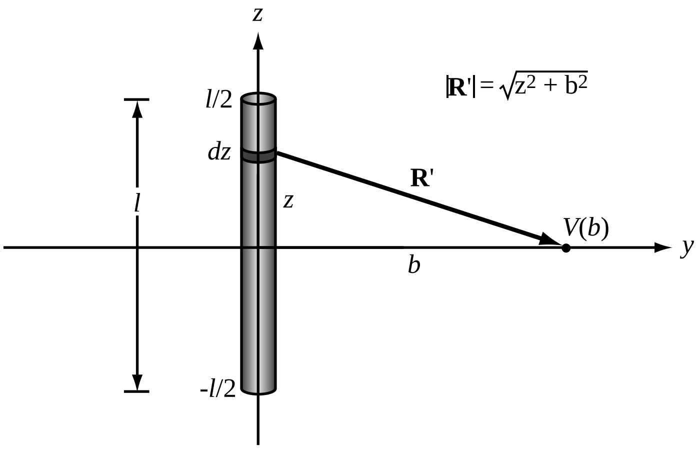


Figure P4.31: Line of charge of length ℓ.

Solution: From Eq. (4.48c), we can find the voltage at a distance b away from a line of charge [Fig. P4.31]:

$$V(b) = \frac{1}{4\pi\varepsilon}\int_{l'}\frac{\rho_l}{R'}\,dl' = \frac{\rho_l}{4\pi\varepsilon}\int_{-l/2}^{l/2}\frac{dz}{\sqrt{z^2+b^2}} = \frac{\rho_l}{4\pi\varepsilon}\ln\left(\frac{l+\sqrt{l^2+4b^2}}{-l+\sqrt{l^2+4b^2}}\right).$$

Problem 4.32 For the electric dipole shown in Fig. 4-13, $d = 1$ cm and $|\mathbf{E}| = 4$ (mV/m) at $R = 1$ m and $\theta = 0°$. Find $\mathbf{E}$ at $R = 2$ m and $\theta = 90°$.

Solution: For $R = 1$ m and $\theta = 0°$, $|\mathbf{E}| = 4$ mV/m, we can solve for q using Eq. (4.56):

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}(\hat{\mathbf{R}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta).$$

Hence,

$$|\mathbf{E}| = \left(\frac{qd}{4\pi\varepsilon_0}\right)2 = 4 \text{ mV/m} \quad \text{at } \theta = 0°,$$

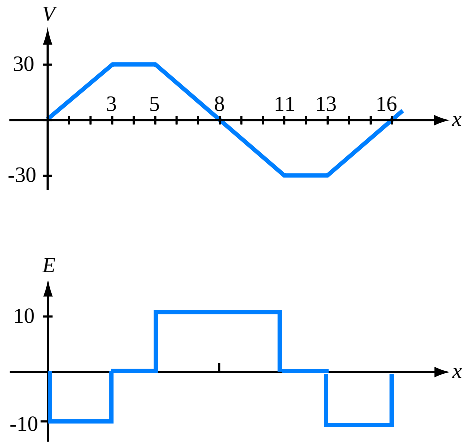
$$q = \frac{10^{-3}\times 8\pi\varepsilon_0}{d} = \frac{10^{-3}\times 8\pi\varepsilon_0}{10^{-2}} = 0.8\pi\varepsilon_0 \quad \text{(C)}.$$

Again using Eq. (4.56) to find $\mathbf{E}$ at $R = 2$ m and $\theta = 90°$, we have

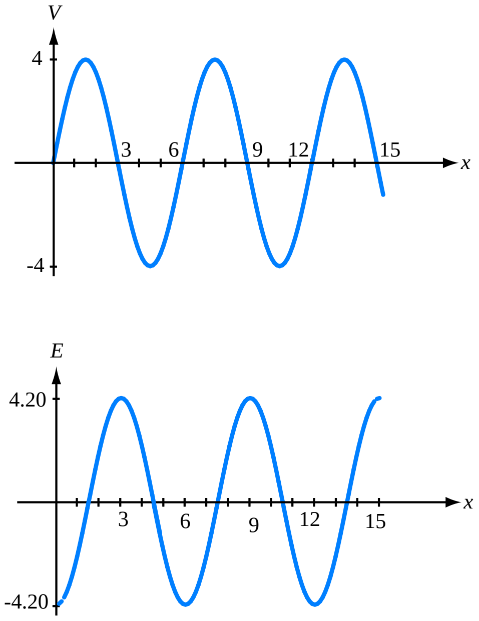
$$\mathbf{E} = \frac{0.8\pi\varepsilon_0\times 10^{-2}}{4\pi\varepsilon_0\times 2^3}(\hat{\mathbf{R}}(0) + \hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\theta}}\,\frac{1}{4} \quad \text{(mV/m)}.$$

Problem 4.33 For each of the following distributions of the electric potential V, sketch the corresponding distribution of $\mathbf{E}$ (in all cases, the vertical axis is in volts and the horizontal axis is in meters):

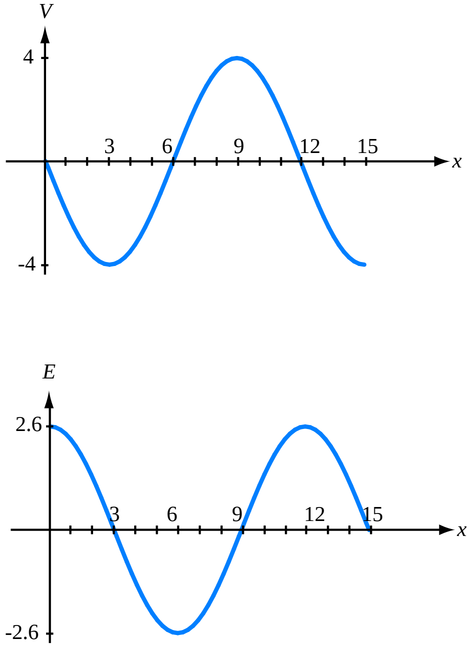
Solution:



(a)



(b)



(c)

Figure P4.33: Electric potential distributions of Problem 4.33.

Problem 4.34 Given the electric field

$$\mathbf{E} = \hat{\mathbf{R}}\,\frac{18}{R^2} \quad \text{(V/m)},$$

find the electric potential of point A with respect to point B where A is at $+2$ m and B at -4 m, both on the z-axis.

Solution:

$$V_{AB} = V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{l}.$$

Along z-direction, $\hat{\mathbf{R}} = \hat{\mathbf{z}}$ and $\mathbf{E} = \hat{\mathbf{z}}\,\frac{18}{z^2}$ for $z \geq 0$, and $\hat{\mathbf{R}} = -\hat{\mathbf{z}}$ and $\mathbf{E} = -\hat{\mathbf{z}}\,\frac{18}{z^2}$ for $z \leq 0$. Hence,

$$V_{AB} = -\int_{-4}^{2} \hat{\mathbf{R}}\frac{18}{z^2} \cdot \hat{\mathbf{z}}\,dz = -\left[\int_{-4}^{0} -\hat{\mathbf{z}}\,\frac{18}{z^2} \cdot \hat{\mathbf{z}}\,dz + \int_{0}^{2} \hat{\mathbf{z}}\,\frac{18}{z^2} \cdot \hat{\mathbf{z}}\,dz\right] = 4 \text{ V}.$$

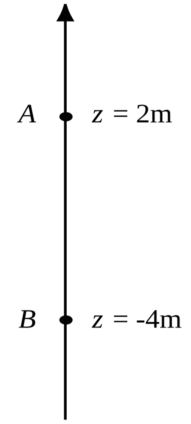


Figure P4.34: Potential between *B* and *A*.

Problem 4.35 An infinitely long line of charge with uniform density $\rho_l = 9$ (nC/m) lies in the x–y plane parallel to the y-axis at $x = 2$ m. Find the potential V_{AB} at point $A(3\text{ m}, 0, 4\text{ m})$ in Cartesian coordinates with respect to point $B(0,0,0)$ by applying the result of Problem 4.30.

Solution: According to Problem 4.30,

$$V = \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{r_2}{r_1}\right)$$

where r_1 and r_2 are the distances of A and B. In this case,

$$r_1 = \sqrt{(3-2)^2 + 4^2} = \sqrt{17}\text{ m},$$
$$r_2 = 2\text{ m}.$$

Hence,

$$V_{AB} = \frac{9 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}} \ln\left(\frac{2}{\sqrt{17}}\right) = -117.09\text{ V}.$$

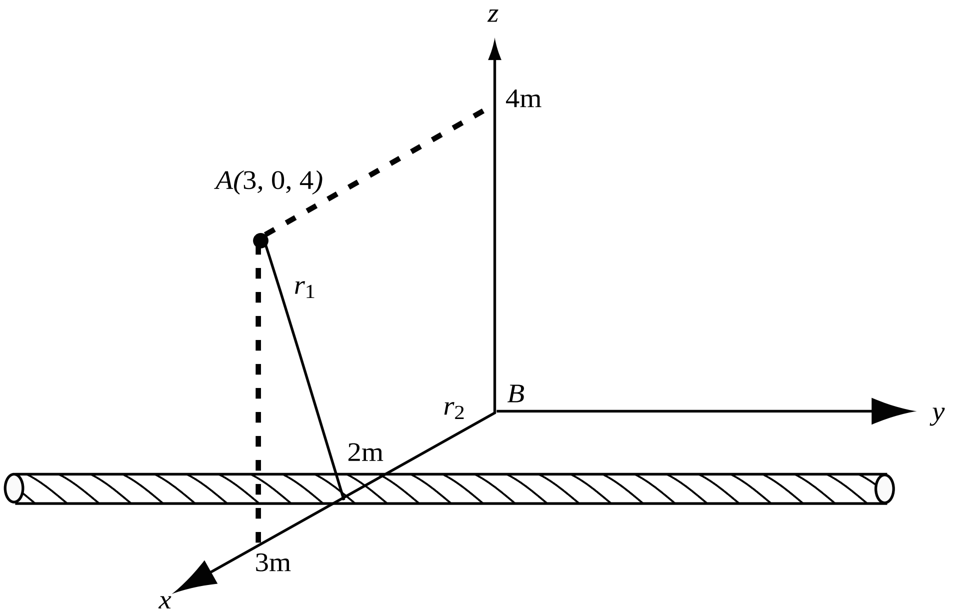


Figure P4.35: Line of charge parallel to y-axis.

Problem 4.36 The x–y plane contains a uniform sheet of charge with $\rho_{s_1} = 0.2$ (nC/m^2) and a second sheet with $\rho_{s_2} = -0.2$ (nC/m^2) occupies the plane $z = 6$ m. Find V_{AB}, V_{BC}, and V_{AC} for $A(0,0,6\text{ m})$, $B(0,0,0)$, and $C(0,-2\text{ m},2\text{ m})$.

Solution: We start by finding the **E** field in the region between the plates. For any point above the x–y plane, $\mathbf{E}_1$ due to the charge on x–y plane is, from Eq. (4.25),

$$\mathbf{E}_1 = \hat{\mathbf{z}}\,\frac{\rho_{s_1}}{2\varepsilon_0}\,.$$

In the region below the top plate, **E** would point downwards for positive ρ_{s_2} on the top plate. In this case, $\rho_{s_2} = -\rho_{s_1}$. Hence,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \hat{\mathbf{z}}\,\frac{\rho_{s_1}}{2\varepsilon_0} - \hat{\mathbf{z}}\,\frac{\rho_{s_2}}{2\varepsilon_0} = \hat{\mathbf{z}}\,\frac{2\rho_{s_1}}{2\varepsilon_0} = \hat{\mathbf{z}}\,\frac{\rho_{s_1}}{\varepsilon_0}\,.$$

Since **E** is along $\hat{\mathbf{z}}$, only change in position along z can result in change in voltage.

$$V_{AB} = -\int_0^6 \hat{\mathbf{z}}\,\frac{\rho_{s_1}}{\varepsilon_0}\cdot\hat{\mathbf{z}}\,dz = -\frac{\rho_{s_1}}{\varepsilon_0}\,z\bigg|_0^6 = -\frac{6\rho_{s_1}}{\varepsilon_0} = -\frac{6\times 0.2\times 10^{-9}}{8.85\times 10^{-12}} = -135.59\text{ V}.$$

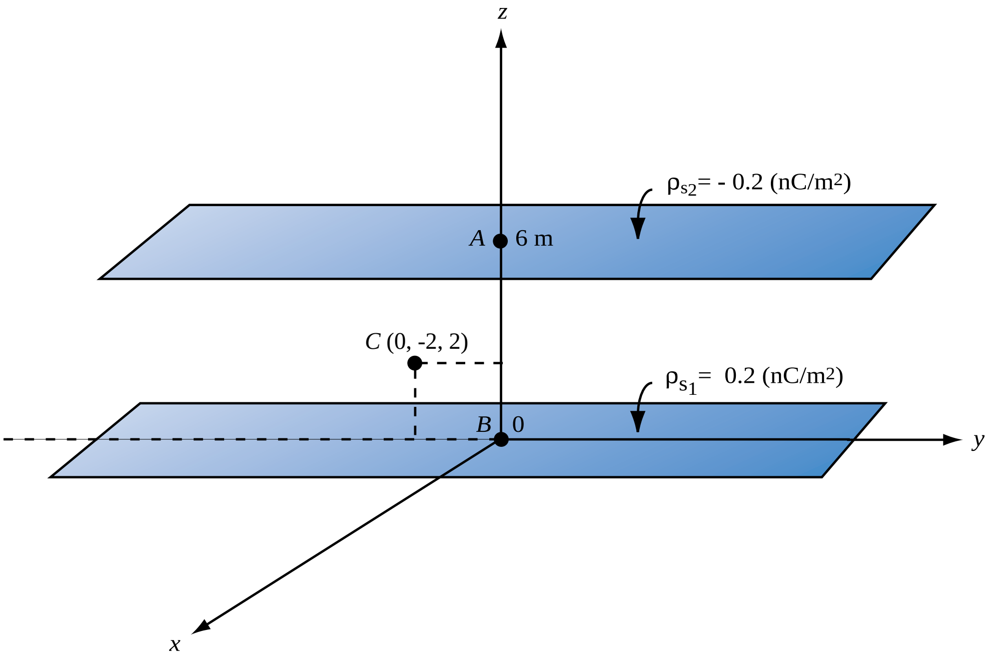


Figure P4.36: Two parallel planes of charge.

The voltage at C depends only on the z-coordinate of C. Hence, with point A being at the lowest potential and B at the highest potential,

$$V_{BC} = \frac{-2}{6} V_{AB} = -\frac{(-135.59)}{3} = 45.20 \text{ V},$$
$$V_{AC} = V_{AB} + V_{BC} = -135.59 + 45.20 = -90.39 \text{ V}.$$

Section 4-7: Conductors

Problem 4.37 A cylindrical bar of silicon has a radius of 4 mm and a length of 8 cm. If a voltage of 5 V is applied between the ends of the bar and $\mu_e = 0.13$ (m^2/V·s), $\mu_h = 0.05$ (m^2/V·s), $N_e = 1.5 \times 10^{16}$ electrons/m^3, and $N_h = N_e$, find

(a) the conductivity of silicon,
(b) the current I flowing in the bar,
(c) the drift velocities $\mathbf{u}_e$ and $\mathbf{u}_h$,
(d) the resistance of the bar, and
(e) the power dissipated in the bar.

Solution:

(a) Conductivity is given in Eq. (4.65),

$$\begin{aligned}\sigma &= (N_e\mu_e + N_h\mu_h)e \\ &= (1.5\times10^{16})(0.13+0.05)(1.6\times10^{-19}) = 4.32\times10^{-4} \quad \text{(S/m)}.\end{aligned}$$

(b) Similarly to Example 4.8, parts b and c,

$$I = JA = \sigma EA = (4.32\times10^{-4})\left(\frac{5\text{V}}{0.08}\right)(\pi(4\times10^{-3})^2) = 1.36 \quad (\mu\text{A}).$$

(c) From Eqs. (4.62a) and (4.62b),

$$\begin{aligned}\mathbf{u}_e &= -\mu_e\mathbf{E} = -(0.13)\left(\frac{5}{0.08}\right)\frac{\mathbf{E}}{|\mathbf{E}|} = -8.125\frac{\mathbf{E}}{|\mathbf{E}|} \quad \text{(m/s)},\\ \mathbf{u}_h &= \mu_h\mathbf{E} = +(0.05)\left(\frac{5}{0.08}\right)\frac{\mathbf{E}}{|\mathbf{E}|} = 3.125\frac{\mathbf{E}}{|\mathbf{E}|} \quad \text{(m/s)}.\end{aligned}$$

(d) To find the resistance, we use what we calculated above,

$$R = \frac{V}{I} = \frac{5\text{V}}{1.36\ \mu\text{A}} = 3.68 \quad (\text{M}\Omega).$$

(e) Power dissipated in the bar is $P = IV = (5\text{V})(1.36\ \mu\text{A}) = 6.8\ (\mu\text{W})$.

Problem 4.38 Repeat Problem 4.37 for a bar of germanium with $\mu_e = 0.4$ (m^2/V·s), $\mu_h = 0.2$ (m^2/V·s), and $N_e = N_h = 2.4\times10^{19}$ electrons or holes/m^3.

Solution:

(a) Conductivity is given in Eq. (4.65),

$$\sigma = (N_e\mu_e + N_u\mu_u)e = (2.4\times10^{19})(0.4+0.2)(1.6\times10^{-19}) = 2.3 \quad \text{(S/m)}.$$

(b) Similarly to Example 4.8, parts b and c,

$$I = JA = \sigma EA = (2.3)\left(\frac{5\text{V}}{0.08}\right)(\pi(4\times10^{-3})^2) = 7.225 \quad \text{(mA)}.$$

(c) From Eqs. (4.62a) and (4.62b),

$$\begin{aligned}\mathbf{u}_e &= -\mu_e\mathbf{E} = -(0.4)\left(\frac{5}{0.08}\right)\frac{\mathbf{E}}{|\mathbf{E}|} = -25\frac{\mathbf{E}}{|\mathbf{E}|} \quad \text{(m/s)},\\ \mathbf{u}_h &= \mu_h\mathbf{E} = (0.2)\left(\frac{5}{0.08}\right) = 12.5\frac{\mathbf{E}}{|\mathbf{E}|} \quad \text{(m/s)}.\end{aligned}$$

(d) To find the resistance, we use what we calculated above,

$$R = \frac{V}{I} = \frac{5\text{V}}{7.225 \text{ mA}} = 0.69 \quad (\text{k}\Omega).$$

(e) Power dissipated in the bar is $P = IV = (5\text{V})(7.225 \text{ mA}) = 36.125$ (mW).

Problem 4.39 A 100-m-long conductor of uniform cross section has a voltage drop of 4 V between its ends. If the density of the current flowing through it is 1.4×10^6 (A/m^2), identify the material of the conductor.

Solution: We know that conductivity characterizes a material:

$$\mathbf{J} = \sigma \mathbf{E}, \qquad 1.4 \times 10^6 \text{ (A/m}^2) = \sigma \left(\frac{4 \text{ (V)}}{100 \text{ (m)}} \right), \qquad \sigma = 3.5 \times 10^7 \quad \text{(S/m)}.$$

From Table B-2, we find that aluminum has $\sigma = 3.5 \times 10^7$ (S/m).

Problem 4.40 A coaxial resistor of length l consists of two concentric cylinders. The inner cylinder has radius a and is made of a material with conductivity σ_1, and the outer cylinder, extending between $r = a$ and $r = b$, is made of a material with conductivity σ_2. If the two ends of the resistor are capped with conducting plates, show that the resistance between the two ends is $R = l/[\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))]$.

Solution: Due to the conducting plates, the ends of the coaxial resistor are each uniform at the same potential. Hence, the electric field everywhere in the resistor will be parallel to the axis of the resistor, in which case the two cylinders can be considered to be two separate resistors in parallel. Then, from Eq. (4.70),

$$\frac{1}{R} = \frac{1}{R_{\text{inner}}} + \frac{1}{R_{\text{outer}}} = \frac{\sigma_1 A_1}{l_1} + \frac{\sigma_2 A_2}{l_2} = \frac{\sigma_1 \pi a^2}{l} + \frac{\sigma_2 \pi (b^2 - a^2)}{l},$$

or

$$R = \frac{l}{\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))} \quad (\Omega).$$

Problem 4.41 Apply the result of Problem 4.40 to find the resistance of a 20-cm-long hollow cylinder (Fig. P4.41) made of carbon with $\sigma = 3 \times 10^4$ (S/m).

Solution: From Problem 4.40, we know that for two concentric cylinders,

$$R = \frac{l}{\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))} \quad (\Omega).$$

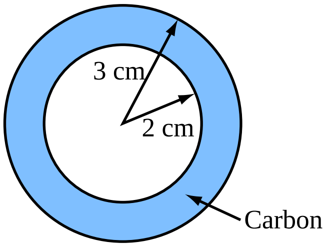


Figure P4.41: Cross section of hollow cylinder of Problem 4.41.

For air $\sigma_1 = 0$ (S/m), $\sigma_2 = 3 \times 10^4$ (S/m); hence,

$$R = \frac{0.2}{3\pi \times 10^4((0.03)^2 - (0.02)^2)} = 4.2 \quad (\text{m}\Omega).$$

Problem 4.42 A 2×10^{-3}-mm-thick square sheet of aluminum has 5 cm × 5 cm faces. Find:

(a) the resistance between opposite edges on a square face, and

(b) the resistance between the two square faces. (See Appendix B for the electrical constants of materials).

Solution:

(a)

$$R = \frac{l}{\sigma A}.$$

For aluminum, $\sigma = 3.5 \times 10^7$ (S/m) [Appendix B].

$$l = 5 \text{ cm}, \qquad A = 5 \text{ cm} \times 2 \times 10^{-3} \text{ mm} = 10 \times 10^{-2} \times 10^{-6} = 1 \times 10^{-7} \text{ m}^2,$$

$$R = \frac{5 \times 10^{-2}}{3.5 \times 10^7 \times 1 \times 10^{-7}} = 14 \quad (\text{m}\Omega).$$

(b) Now, $l = 2 \times 10^{-3}$ mm and $A = 5 \text{ cm} \times 5 \text{ cm} = 2.5 \times 10^{-3} \text{ m}^2$.

$$R = \frac{2 \times 10^{-6}}{3.5 \times 10^7 \times 2.5 \times 10^{-3}} = 22.8 \text{ p}\Omega.$$

Section 4-9: Boundary Conditions

Problem 4.43 With reference to Fig. 4-19, find $\mathbf{E}_1$ if $\mathbf{E}_2 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}2$ (V/m), $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 18\varepsilon_0$, and the boundary has a surface charge density $\rho_s = 3.54 \times 10^{-11}$ (C/m^2). What angle does $\mathbf{E}_2$ make with the z-axis?

Solution: We know that $\mathbf{E}_{1t} = \mathbf{E}_{2t}$ for any 2 media. Hence, $\mathbf{E}_{1t} = \mathbf{E}_{2t} = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2$. Also, $(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}} = \rho_s$ (from Table 4.3). Hence, $\varepsilon_1(\mathbf{E}_1 \cdot \hat{\mathbf{n}}) - \varepsilon_2(\mathbf{E}_2 \cdot \hat{\mathbf{n}}) = \rho_s$, which gives

$$E_{1z} = \frac{\rho_s + \varepsilon_2 E_{2z}}{\varepsilon_1} = \frac{3.54 \times 10^{-11}}{2\varepsilon_0} + \frac{18(2)}{2} = \frac{3.54 \times 10^{-11}}{2 \times 8.85 \times 10^{-12}} + 18 = 20 \quad \text{(V/m)}.$$

Hence, $\mathbf{E}_1 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}20$ (V/m). Finding the angle $\mathbf{E}_2$ makes with the z-axis:

$$\mathbf{E}_2 \cdot \hat{\mathbf{z}} = |\mathbf{E}_2| \cos\theta, \qquad 2 = \sqrt{9+4+4}\cos\theta, \qquad \theta = \cos^{-1}\left(\frac{2}{\sqrt{17}}\right) = 61^\circ.$$

Problem 4.44 An infinitely long conducting cylinder of radius a has a surface charge density ρ_s. The cylinder is surrounded by a dielectric medium with $\varepsilon_r = 4$ and contains no free charges. If the tangential component of the electric field in the region $r \geq a$ is given by $\mathbf{E}_t = -\hat{\boldsymbol{\phi}} \cos^2\phi / r^2$, find ρ_s.

Solution: Let the conducting cylinder be medium 1 and the surrounding dielectric medium be medium 2. In medium 2,

$$\mathbf{E}_2 = \hat{\mathbf{r}} E_r - \hat{\boldsymbol{\phi}} \frac{1}{r^2} \cos^2\phi,$$

with E_r, the normal component of $\mathbf{E}_2$, unknown. The surface charge density is related to E_r. To find E_r, we invoke Gauss's law in medium 2:

$$\nabla \cdot \mathbf{D}_2 = 0,$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(-\frac{1}{r^2}\cos^2\phi\right) = 0,$$

which leads to

$$\frac{\partial}{\partial r}(rE_r) = \frac{\partial}{\partial\phi}\left(\frac{1}{r^2}\cos^2\phi\right) = -\frac{2}{r^2}\sin\phi\cos\phi.$$

Integrating both sides with respect to r,

$$\int \frac{\partial}{\partial r}(rE_r)\, dr = -2\sin\phi\cos\phi \int \frac{1}{r^2}\, dr$$

$$rE_r = \frac{2}{r}\sin\phi\cos\phi,$$

or

$$E_r = \frac{2}{r^2}\sin\phi\cos\phi.$$

Hence,

$$\mathbf{E}_2 = \hat{\mathbf{r}}\,\frac{2}{r^2}\sin\phi\cos\phi - \hat{\boldsymbol{\phi}}\,\frac{1}{r^2}\cos^2\phi.$$

According to Eq. (4.93),

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s,$$

where $\hat{\mathbf{n}}_2$ is the normal to the boundary and points away from medium 1. Hence, $\hat{\mathbf{n}}_2 = \hat{\mathbf{r}}$. Also, $\mathbf{D}_1 = 0$ because the cylinder is a conductor. Consequently,

$$\begin{aligned}
\rho_s &= -\hat{\mathbf{r}} \cdot \mathbf{D}_2\big|_{r=a} \\
&= -\hat{\mathbf{r}} \cdot \varepsilon_2 \mathbf{E}_2\big|_{r=a} \\
&= -\hat{\mathbf{r}} \cdot \varepsilon_r\varepsilon_0 \left[\hat{\mathbf{r}}\,\frac{2}{r^2}\sin\phi\cos\phi - \hat{\boldsymbol{\phi}}\,\frac{1}{r^2}\cos^2\phi\right]\Bigg|_{r=a} \\
&= -\frac{8\varepsilon_0}{a^2}\sin\phi\cos\phi \quad (\text{C/m}^2).
\end{aligned}$$

Problem 4.45 A 2-cm conducting sphere is embedded in a charge-free dielectric medium with $\varepsilon_{2r} = 9$. If $\mathbf{E}_2 = \hat{\mathbf{R}}3\cos\theta - \hat{\boldsymbol{\theta}}3\sin\theta$ (V/m) in the surrounding region, find the charge density on the sphere's surface.

Solution: According to Eq. (4.93),

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s.$$

In the present case, $\hat{\mathbf{n}}_2 = \hat{\mathbf{R}}$ and $\mathbf{D}_1 = 0$. Hence,

$$\begin{aligned}
\rho_s &= -\hat{\mathbf{R}} \cdot \mathbf{D}_2\big|_{r=2\text{ cm}} \\
&= -\hat{\mathbf{R}} \cdot \varepsilon_2(\hat{\mathbf{R}}3\cos\theta - \hat{\boldsymbol{\theta}}3\sin\theta) \\
&= -27\varepsilon_0\cos\theta \quad (\text{C/m}^2).
\end{aligned}$$

Problem 4.46 If $\mathbf{E} = \hat{\mathbf{R}}150$ (V/m) at the surface of a 5-cm conducting sphere centered at the origin, what is the total charge Q on the sphere's surface?

Solution: From Table 4-3, $\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$. $\mathbf{E}_2$ inside the sphere is zero, since we assume it is a perfect conductor. Hence, for a sphere with surface area $S = 4\pi a^2$,

$$D_{1R} = \rho_s, \qquad E_{1R} = \frac{\rho_s}{\varepsilon_0} = \frac{Q}{S\varepsilon_0},$$

$$Q = E_R S\varepsilon_0 = (150)4\pi(0.05)^2\varepsilon_0 = \frac{3\pi\varepsilon_0}{2} \quad \text{(C)}.$$

Problem 4.47 Figure 4-34(a) (P4.47) shows three planar dielectric slabs of equal thickness but with different dielectric constants. If $\mathbf{E}_0$ in air makes an angle of 45° with respect to the z-axis, find the angle of $\mathbf{E}$ in each of the other layers.

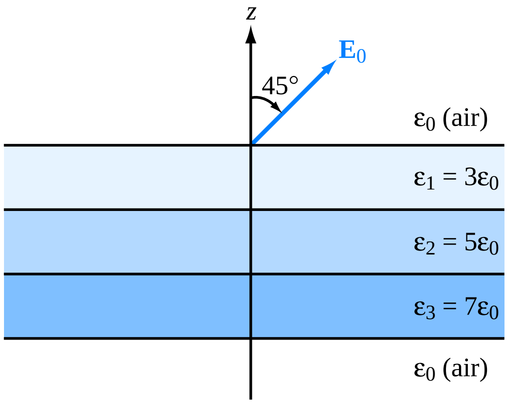

z
$\mathbf{E}_0$
45°
ε_0 (air)
$\varepsilon_1 = 3\varepsilon_0$
$\varepsilon_2 = 5\varepsilon_0$
$\varepsilon_3 = 7\varepsilon_0$
ε_0 (air)

Figure P4.47: Dielectric slabs in Problem 4.47.

Solution: Labeling the upper air region as region 0 and using Eq. (4.99),

$$\theta_1 = \tan^{-1}\left(\frac{\varepsilon_1}{\varepsilon_0}\tan\theta_0\right) = \tan^{-1}(3\tan 45°) = 71.6°,$$

$$\theta_2 = \tan^{-1}\left(\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right) = \tan^{-1}\left(\frac{5}{3}\tan 71.6°\right) = 78.7°,$$

$$\theta_3 = \tan^{-1}\left(\frac{\varepsilon_3}{\varepsilon_2}\tan\theta_2\right) = \tan^{-1}\left(\frac{7}{5}\tan 78.7°\right) = 81.9°.$$

In the lower air region, the angle is again 45°.

Sections 4-10 and 4-11: Capacitance and Electrical Energy

Problem 4.48 Determine the force of attraction in a parallel-plate capacitor with $A = 5$ cm^2, $d = 2$ cm, and $\varepsilon_r = 4$ if the voltage across it is 50 V.

Solution: From Eq. (4.131),

$$\mathbf{F} = -\hat{\mathbf{z}}\,\frac{\varepsilon A|\mathbf{E}|^2}{2} = -\hat{\mathbf{z}}\,2\varepsilon_0(5\times10^{-4})\left(\frac{50}{0.02}\right)^2 = -\hat{\mathbf{z}}\,55.3\times10^{-9} \quad \text{(N)}.$$

Problem 4.49 Dielectric breakdown occurs in a material whenever the magnitude of the field $\mathbf{E}$ exceeds the dielectric strength anywhere in that material. In the coaxial capacitor of Example 4-12,

(a) At what value of r is $|E|$ maximum?

(b) What is the breakdown voltage if $a = 1$ cm, $b = 2$ cm, and the dielectric material is mica with $\varepsilon_r = 6$?

Solution:

(a) From Eq. (4.114), $\mathbf{E} = -\hat{\mathbf{r}}\rho_l/2\pi\varepsilon r$ for $a < r < b$. Thus, it is evident that $|\mathbf{E}|$ is maximum at $r = a$.

(b) The dielectric breaks down when $|\mathbf{E}| > 200$ (MV/m) (see Table 4-2), or

$$|\mathbf{E}| = \frac{\rho_l}{2\pi\varepsilon r} = \frac{\rho_l}{2\pi(6\varepsilon_0)(10^{-2})} = 200 \quad \text{(MV/m)},$$

which gives $\rho_l = (200\ \text{MV/m})(2\pi)6(8.854\times10^{-12})(0.01) = 667.6\ (\mu\text{C/m})$.

From Eq. (4.115), we can find the voltage corresponding to that charge density,

$$V = \frac{\rho_l}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right) = \frac{(667.6\,\mu\text{C/m})}{12\pi(8.854\times10^{-12}\ \text{F/m})}\ln(2) = 1.39 \quad \text{(MV)}.$$

Thus, $V = 1.39$ (MV) is the breakdown voltage for this capacitor.

Problem 4.50 An electron with charge $Q_e = -1.6\times10^{-19}$ C and mass $m_e = 9.1\times10^{-31}$ kg is injected at a point adjacent to the negatively charged plate in the region between the plates of an air-filled parallel-plate capacitor with separation of 1 cm and rectangular plates each 10 cm^2 in area Fig. 4-33 (P4.50). If the voltage across the capacitor is 10 V, find

(a) the force acting on the electron,

(b) the acceleration of the electron, and

(c) the time it takes the electron to reach the positively charged plate, assuming that it starts from rest.

Solution:

(a) The electric force acting on a charge Q_e is given by Eq. (4.14) and the electric field in a capacitor is given by Eq. (4.112). Combining these two relations, we have

$$F = Q_e E = Q_e\frac{V}{d} = -1.6\times10^{-19}\,\frac{10}{0.01} = -1.6\times10^{-16} \quad \text{(N)}.$$

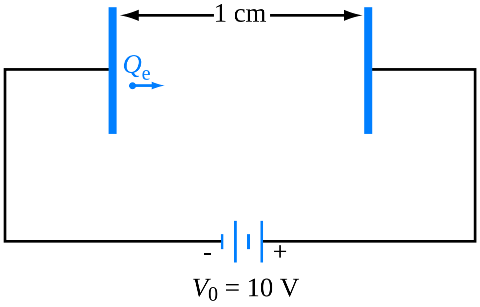


Figure P4.50: Electron between charged plates of Problem 4.50.

The force is directed from the negatively charged plate towards the positively charged plate.

(b)

$$a = \frac{F}{m} = \frac{1.6 \times 10^{-16}}{9.1 \times 10^{-31}} = 1.76 \times 10^{14} \quad (\text{m/s}^2).$$

(c) The electron does not get fast enough at the end of its short trip for relativity to manifest itself; classical mechanics is adequate to find the transit time. From classical mechanics, $d = d_0 + u_0 t + \frac{1}{2}at^2$, where in the present case the start position is $d_0 = 0$, the total distance traveled is $d = 1$ cm, the initial velocity $u_0 = 0$, and the acceleration is given by part (b). Solving for the time t,

$$t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2 \times 0.01}{1.76 \times 10^{14}}} = 10.7 \times 10^{-9} \text{ s} = 10.7 \quad (\text{ns}).$$

Problem 4.51 In a dielectric medium with $\varepsilon_r = 4$, the electric field is given by

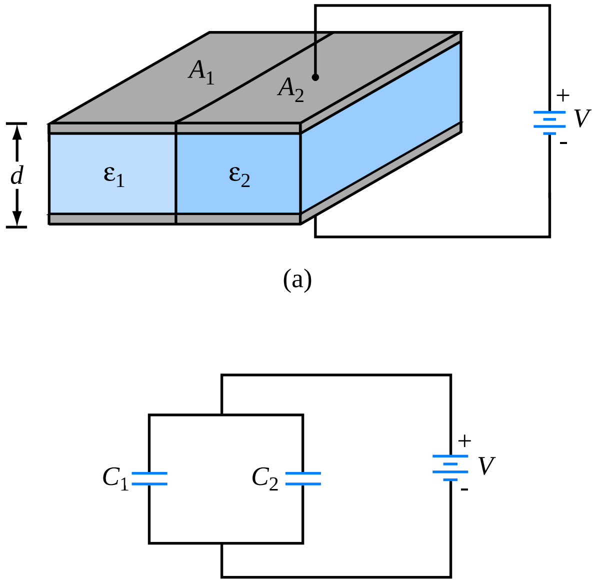
$$\mathbf{E} = \hat{\mathbf{x}}(x^2 + 2z) + \hat{\mathbf{y}}x^2 - \hat{\mathbf{z}}(y + z) \quad (\text{V/m}).$$

Calculate the electrostatic energy stored in the region $-1 \text{ m} \le x \le 1 \text{ m}$, $0 \le y \le 2$ m, and $0 \le z \le 3$ m.

Solution: Electrostatic potential energy is given by Eq. (4.124),

$$
\begin{aligned}
W_\mathrm{e} &= \frac{1}{2}\int_{\mathcal{V}} \varepsilon|\mathbf{E}|^2\,d\mathcal{V} = \frac{\varepsilon}{2}\int_{z=0}^{3}\int_{y=0}^{2}\int_{x=-1}^{1}[(x^2+2z)^2+x^4+(y+z)^2]\,dx\,dy\,dz \\
&= \frac{4\varepsilon_0}{2}\left(\left(\left(\frac{2}{5}x^5yz+\frac{2}{3}z^2x^3y+\frac{4}{3}z^3xy+\frac{1}{12}(y+z)^4x\right)\Bigg|_{x=-1}^{1}\right)\Bigg|_{y=0}^{2}\right)\Bigg|_{z=0}^{3} \\
&= \frac{4\varepsilon_0}{2}\left(\frac{1304}{5}\right) = 4.62\times10^{-9} \quad \text{(J)}.
\end{aligned}
$$

Problem 4.52 Figure 4-34a (P4.52(a)) depicts a capacitor consisting of two parallel, conducting plates separated by a distance d. The space between the plates contains two adjacent dielectrics, one with permittivity ε_1 and surface area A_1 and another with ε_2 and A_2. The objective of this problem is to show that the capacitance C of the configuration shown in Fig. 4-34a (P4.52(a)) is equivalent to two capacitances in parallel, as illustrated in Fig. 4-34b (P4.52(b)), with



(a)

(b)

Figure P4.52: (a) Capacitor with parallel dielectric section, and (b) equivalent circuit.

$$C = C_1 + C_2, \tag{4.132}$$

where

$$C_1 = \frac{\varepsilon_1 A_1}{d}, \tag{4.133}$$

$$C_2 = \frac{\varepsilon_2 A_2}{d}. \tag{4.134}$$

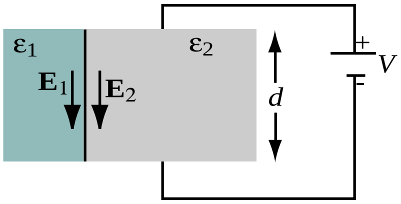
To this end, you are asked to proceed as follows:

(a) Find the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ in the two dielectric layers.

(b) Calculate the energy stored in each section and use the result to calculate C_1 and C_2.

(c) Use the total energy stored in the capacitor to obtain an expression for C. Show that Eq. (4.132) is indeed a valid result.

Solution:



(c)

Figure P4.52: (c) Electric field inside of capacitor.

(a) Within each dielectric section, **E** will point from the plate with positive voltage to the plate with negative voltage, as shown in Fig. P4-52(c). From $V = Ed$,

$$E_1 = E_2 = \frac{V}{d}.$$

(b)

$$W_{e_1} = \frac{1}{2}\varepsilon_1 E_1^2 \cdot \mathcal{V} = \frac{1}{2}\varepsilon_1 \frac{V^2}{d^2} \cdot A_1 d = \frac{1}{2}\varepsilon_1 V^2 \frac{A_1}{d}.$$

But, from Eq. (4.121),

$$W_{e_1} = \frac{1}{2} C_1 V^2.$$

Hence $C_1 = \varepsilon_1 \dfrac{A_1}{d}$. Similarly, $C_2 = \varepsilon_2 \dfrac{A_2}{d}$.

(c) Total energy is

$$W_e = W_{e_1} + W_{e_2} = \frac{1}{2}\frac{V^2}{d}(\varepsilon_1 A_1 + \varepsilon_2 A_2) = \frac{1}{2} C V^2.$$

Hence,

$$C = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} = C_1 + C_2.$$

Problem 4.53 Use the result of Problem 4.52 to determine the capacitance for each of the following configurations:

(a) conducting plates are on top and bottom faces of rectangular structure in Fig. 4-35(a) (P4.53(a)),

(b) conducting plates are on front and back faces of structure in Fig. 4-35(a) (P4.53(a)),

(c) conducting plates are on top and bottom faces of the cylindrical structure in Fig. 4-35(b) (P4.53(b)).

Solution:

(a) The two capacitors share the same voltage; hence they are in parallel.

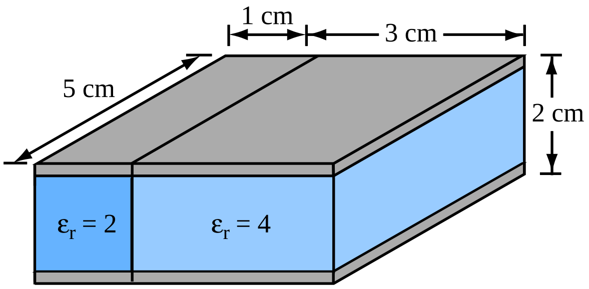
$$\begin{aligned}
C_1 &= \varepsilon_1 \frac{A_1}{d} = 2\varepsilon_0 \frac{(5\times 1)\times 10^{-4}}{2\times 10^{-2}} = 5\varepsilon_0 \times 10^{-2}, \\
C_2 &= \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(5\times 3)\times 10^{-4}}{2\times 10^{-2}} = 30\varepsilon_0 \times 10^{-2}, \\
C &= C_1 + C_2 = (5\varepsilon_0 + 30\varepsilon_0)\times 10^{-2} = 0.35\varepsilon_0 = 3.1\times 10^{-12} \text{ F.}
\end{aligned}$$

(b)

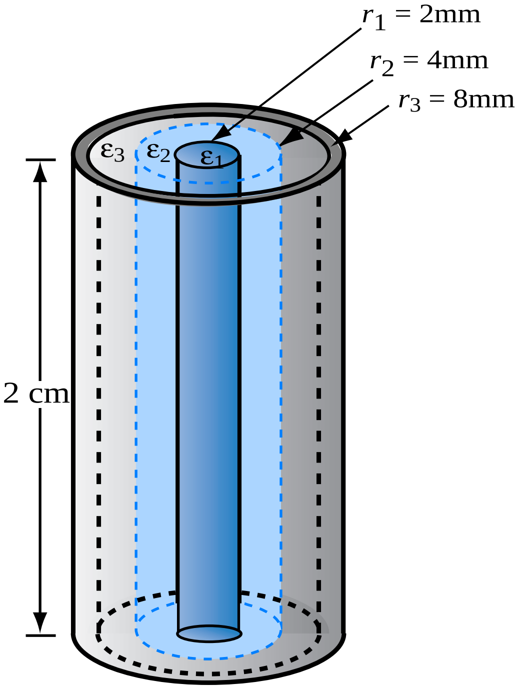
$$\begin{aligned}
C_1 &= \varepsilon_1 \frac{A_1}{d} = 2\varepsilon_0 \frac{(2\times 1)\times 10^{-4}}{5\times 10^{-2}} = 0.8\,\varepsilon_0 \times 10^{-2}, \\
C_2 &= \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(3\times 2)\times 10^{-4}}{5\times 10^{-2}} = \frac{24}{5}\,\varepsilon_0 \times 10^{-2}, \\
C &= C_1 + C_2 = 0.5\times 10^{-12} \text{ F.}
\end{aligned}$$

(c)

$$\begin{aligned}
C_1 &= \varepsilon_1 \frac{A_1}{d} = 8\varepsilon_0 \frac{(\pi r_1^2)}{2\times 10^{-2}} = \frac{4\pi\varepsilon_0}{10^{-2}}\,(2\times 10^{-3})^2 = 0.04\times 10^{-12} \text{ F,} \\
C_2 &= \varepsilon_2 \frac{A_2}{d} = 4\varepsilon_0 \frac{(\pi(r_2^2 - r_1^2))}{2\times 10^{-2}} = \frac{2\pi\varepsilon_0}{10^{-2}}\,[(4\times 10^{-3})^2 - (2\times 10^{-3})^2] = 0.06\times 10^{-12} \text{ F,} \\
C_3 &= \varepsilon_3 \frac{A_3}{d} = 2\varepsilon_0 \frac{(\pi(r_3^2 - r_2^2))}{2\times 10^{-2}} = \frac{\pi\varepsilon_0}{10^{-2}}\,[(8\times 10^{-3})^2 - (4\times 10^{-3})^2] = 0.12\times 10^{-12} \text{ F,} \\
C &= C_1 + C_2 + C_3 = 0.22\times 10^{-12} \text{ F.}
\end{aligned}$$



(a)



$\varepsilon_1 = 8\varepsilon_0$; $\varepsilon_2 = 4\varepsilon_0$; $\varepsilon_3 = 2\varepsilon_0$

(b)

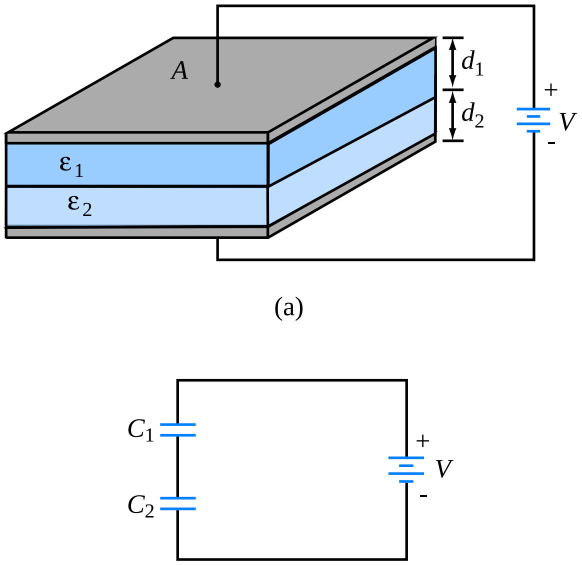
Figure P4.53: Dielectric sections for Problems 4.53 and 4.55.

Problem 4.54 The capacitor shown in Fig. 4-36 (P4.54) consists of two parallel dielectric layers. We wish to use energy considerations to show that the equivalent capacitance of the overall capacitor, C, is equal to the series combination of the capacitances of the individual layers, C_1 and C_2, namely

$$C = \frac{C_1 C_2}{C_1 + C_2}, \tag{4.136}$$

where

$$C_1 = \varepsilon_1 \frac{A}{d_1}, \qquad C_2 = \varepsilon_2 \frac{A}{d_2}.$$



(a)

(b)

Figure P4.54: (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.54).

- **(a)** Let V_1 and V_2 be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields E_1 and E_2? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for E_1 and E_2 in terms of ε_1, ε_2, V, and the indicated dimensions of the capacitor.
- **(b)** Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for C.

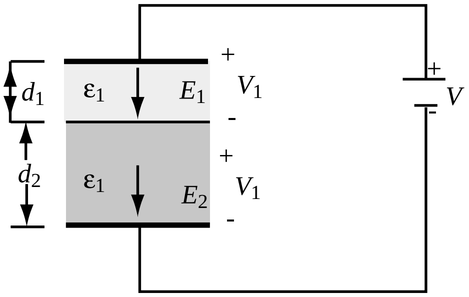


Figure P4.54: (c) Electric fields inside of capacitor.

(c) Show that C is given by Eq. (4.136).

Solution:

(a) If V_1 is the voltage across the top layer and V_2 across the bottom layer, then

$$V = V_1 + V_2,$$

and

$$E_1 = \frac{V_1}{d_1}, \qquad E_2 = \frac{V_2}{d_2}.$$

According to boundary conditions, the normal component of **D** is continuous across the boundary (in the absence of surface charge). This means that at the interface between the two dielectric layers,

$$D_{1\mathrm{n}} = D_{2\mathrm{n}}$$

or

$$\varepsilon_1 E_1 = \varepsilon_2 E_2.$$

Hence,

$$V = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\varepsilon_1 E_1}{\varepsilon_2} d_2,$$

which can be solved for E_1:

$$E_1 = \frac{V}{d_1 + \dfrac{\varepsilon_1}{\varepsilon_2} d_2}.$$

Similarly,

$$E_2 = \frac{V}{d_2 + \dfrac{\varepsilon_2}{\varepsilon_1} d_1}.$$

(b)

$$W_{e_1} = \frac{1}{2}\varepsilon_1 E_1^2 \cdot \mathcal{V}_1 = \frac{1}{2}\varepsilon_1 \left(\frac{V}{d_1 + \dfrac{\varepsilon_1}{\varepsilon_2} d_2} \right)^2 \cdot A d_1 = \frac{1}{2} V^2 \left[\frac{\varepsilon_1 \varepsilon_2^2 A d_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} \right],$$

$$W_{e_2} = \frac{1}{2}\varepsilon_2 E_2^2 \cdot \mathcal{V}_2 = \frac{1}{2}\varepsilon_2 \left(\frac{V}{d_2 + \dfrac{\varepsilon_2}{\varepsilon_1} d_1} \right)^2 \cdot A d_2 = \frac{1}{2} V^2 \left[\frac{\varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right],$$

$$W_e = W_{e_1} + W_{e_2} = \frac{1}{2} V^2 \left[\frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2} \right].$$

But $W_e = \frac{1}{2} C V^2$, hence,

$$C = \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \varepsilon_1 \varepsilon_2 A \frac{(\varepsilon_2 d_1 + \varepsilon_1 d_2)}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \frac{\varepsilon_1 \varepsilon_2 A}{\varepsilon_2 d_1 + \varepsilon_1 d_2}.$$

(c) Multiplying numerator and denominator of the expression for C by $A/d_1 d_2$, we have

$$C = \frac{\dfrac{\varepsilon_1 A}{d_1} \cdot \dfrac{\varepsilon_2 A}{d_2}}{\dfrac{\varepsilon_1 A}{d_1} + \dfrac{\varepsilon_2 A}{d_2}} = \frac{C_1 C_2}{C_1 + C_2},$$

where

$$C_1 = \frac{\varepsilon_1 A}{d_1}, \qquad C_2 = \frac{\varepsilon_2 A}{d_2}.$$

Problem 4.55 Use the expressions given in Problem 4.54 to determine the capacitance for the configurations in Fig. 4.35(a) (P4.55) when the conducting plates are placed on the right and left faces of the structure.

Solution:

$$C_1 = \varepsilon_1 \frac{A}{d_1} = 2\varepsilon_0 \frac{(2 \times 5) \times 10^{-4}}{1 \times 10^{-2}} = 20\varepsilon_0 \times 10^{-2} = 1.77 \times 10^{-12} \text{ F},$$

$$C_2 = \varepsilon_2 \frac{A}{d_2} = 4\varepsilon_0 \frac{(2 \times 5) \times 10^{-4}}{3 \times 10^{-2}} = 1.18 \times 10^{-12} \text{ F},$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{1.77 \times 1.18}{1.77 + 1.18} \times 10^{-12} = 0.71 \times 10^{-12} \text{ F}.$$

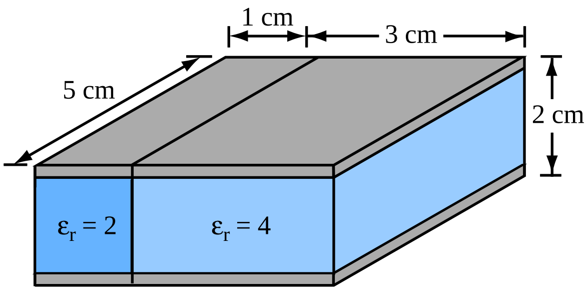


Figure P4.55: Dielectric section for Problem 4.55.

Section 4-12: Image Method

Problem 4.56 With reference to Fig. 4-37 (P4.56), charge Q is located at a distance d above a grounded half-plane located in the x–y plane and at a distance d from another grounded half-plane in the x–z plane. Use the image method to

(a) establish the magnitudes, polarities, and locations of the images of charge Q with respect to each of the two ground planes (as if each is infinite in extent), and

(b) then find the electric potential and electric field at an arbitrary point $P(0,y,z)$.

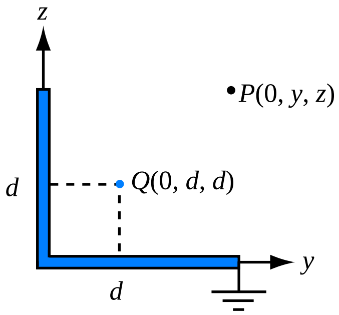


Figure P4.56: Charge Q next to two perpendicular, grounded, conducting half planes.

Solution:

(a) The original charge has magnitude and polarity $+Q$ at location $(0,d,d)$. Since the negative y-axis is shielded from the region of interest, there might as well be a conducting half-plane extending in the $-y$ direction as well as the $+y$ direction. This ground plane gives rise to an image charge of magnitude and polarity $-Q$ at location

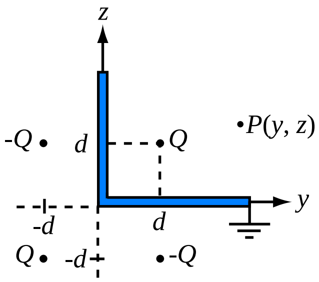


Figure P4.56: (a) Image charges.

$(0,d,-d)$. In addition, since charges exist on the conducting half plane in the $+z$ direction, an image of this conducting half plane also appears in the $-z$ direction. This ground plane in the x-z plane gives rise to the image charges of $-Q$ at $(0,-d,d)$ and $+Q$ at $(0,-d,-d)$.

(b) Using Eq. (4.47) with $N = 4$,

$$\begin{aligned}
V(x,y,z) &= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)|}-\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)|}\right.\\
&\qquad\left.+\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)|}-\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)|}\right)\\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}}-\frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}}\right.\\
&\qquad\left.+\frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}}-\frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right)\\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2+y^2-2yd+z^2-2zd+2d^2}}\right.\\
&\qquad-\frac{1}{\sqrt{x^2+y^2+2yd+z^2-2zd+2d^2}}\\
&\qquad+\frac{1}{\sqrt{x^2+y^2+2yd+z^2+2zd+2d^2}}\\
&\qquad\left.-\frac{1}{\sqrt{x^2+y^2-2yd+z^2+2zd+2d^2}}\right) \quad \text{(V)}.
\end{aligned}$$

From Eq. (4.51),

$$\begin{aligned}
\mathbf{E} &= -\nabla V \\
&= \frac{Q}{4\pi\varepsilon}\left(\nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}} - \nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}}\right. \\
&\qquad \left. +\nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}} - \nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right) \\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)}{(x^2+(y-d)^2+(z-d)^2)^{3/2}} - \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)}{(x^2+(y+d)^2+(z-d)^2)^{3/2}}\right. \\
&\qquad \left. +\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)}{(x^2+(y+d)^2+(z+d)^2)^{3/2}} - \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)}{(x^2+(y-d)^2+(z+d)^2)^{3/2}}\right) \quad \text{(V/m)}.
\end{aligned}$$

Problem 4.57 Conducting wires above a conducting plane carry currents I_1 and I_2 in the directions shown in Fig. 4-38 (P4.57). Keeping in mind that the direction of a current is defined in terms of the movement of positive charges, what are the directions of the image currents corresponding to I_1 and I_2?

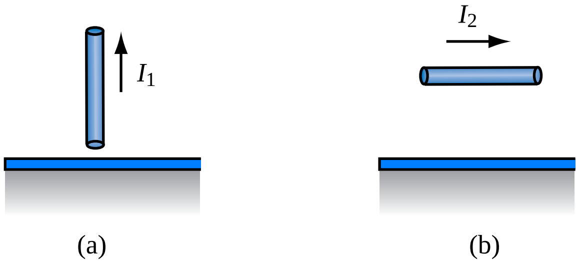


Figure P4.57: Currents above a conducting plane (Problem 4.57).

Solution:

(a) In the image current, movement of negative charges downward = movement of positive charges upward. Hence, image of I_1 is same as I_1.

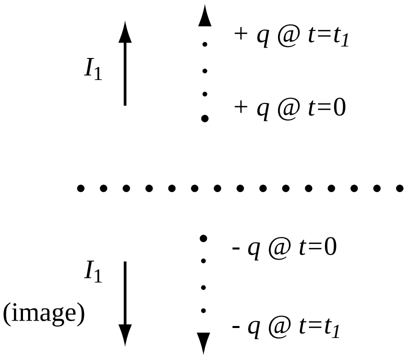


Figure P4.57: (a) Solution for part (a).

(b) In the image current, movement of negative charges to right = movement of positive charges to left.

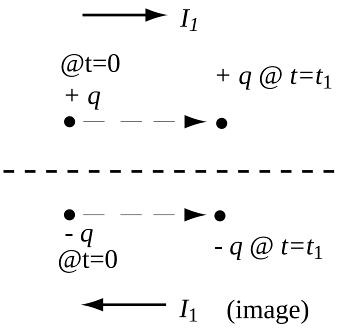


Figure P4.57: (b) Solution for part (b).

Problem 4.58 Use the image method to find the capacitance per unit length of an infinitely long conducting cylinder of radius a situated at a distance d from a parallel conducting plane, as shown in Fig. 4-39 (P4.58).

Solution: Let us distribute charge ρ_l (C/m) on the conducting cylinder. Its image cylinder at $z = -d$ will have charge density $-\rho_l$.

For the line at $z = d$, the electric field at any point z (at a distance of $d - z$ from the center of the cylinder) is, from Eq. (4.33),

$$\mathbf{E}_1 = -\hat{\mathbf{z}} \frac{\rho_l}{2\pi\varepsilon_0 (d - z)}$$

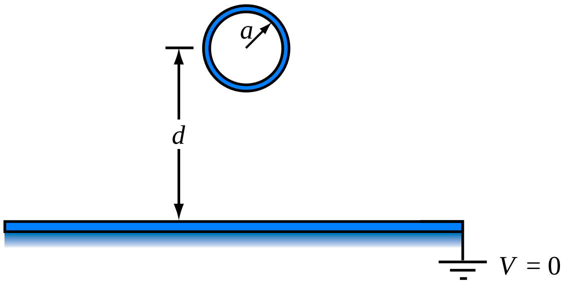


Figure P4.58: Conducting cylinder above a conducting plane (Problem 4.58).

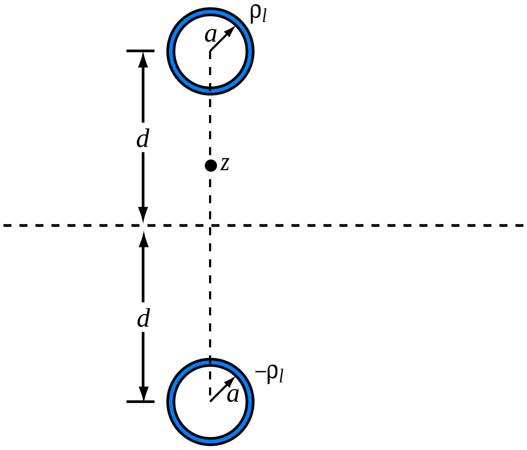


Figure P4.58: (a) Cylinder and its image.

where $-\hat{\mathbf{z}}$ is the direction away from the cylinder. Similarly for the image cylinder at distance $(d+z)$ and carrying charge $-\rho_l$,

$$\mathbf{E}_2 = \hat{\mathbf{z}}\,\frac{(-\rho_l)}{2\pi\varepsilon_0(d+z)} = -\hat{\mathbf{z}}\,\frac{\rho_l}{2\pi\varepsilon_0(d+z)}\,.$$

The potential difference between the cylinders is obtained by integrating the total electric field from $z=-(d-a)$ to $z=(d-a)$:

$$\begin{aligned} V &= -\int_2^1 (\mathbf{E}_1+\mathbf{E}_2)\cdot\hat{\mathbf{z}}\,dz \\ &= -\int_{-(d-a)}^{d-a} -\hat{\mathbf{z}}\,\frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{d-z}+\frac{1}{d+z}\right)\cdot\hat{\mathbf{z}}\,dz \end{aligned}$$

$$
\begin{aligned}
&= \frac{\rho_l}{2\pi\varepsilon_0} \int_{-(d-a)}^{d-a} \left(\frac{1}{d-z} + \frac{1}{d+z} \right) dz \\
&= \frac{\rho_l}{2\pi\varepsilon_0} \left[-\ln(d-z) + \ln(d+z) \right]_{-(d-a)}^{d-a} \\
&= \frac{\rho_l}{2\pi\varepsilon_0} \left[-\ln(a) + \ln(2d-a) + \ln(2d-a) - \ln(a) \right] \\
&= \frac{\rho_l}{\pi\varepsilon_0} \ln\left(\frac{2d-a}{a} \right).
\end{aligned}
$$

For a length L, $Q = \rho_l L$ and

$$C = \frac{Q}{V} = \frac{\rho_l L}{(\rho_l/\pi\varepsilon_0)\ln[(2d-a)/a]},$$

and the capacitance per unit length is

$$C' = \frac{C}{L} = \frac{\pi\varepsilon_0}{\ln[(2d/a)-1]} \quad \text{(C/m)}.$$

Problem 4.59 A circular beam of charge of radius a consists of electrons moving with a constant speed u along the $+z$ direction. The beam's axis is coincident with the z-axis and the electron charge density is given by

$$\rho_v = -cr^2 \quad (\text{c/m}^3)$$

where c is a constant and r is the radial distance from the axis of the beam.

(a) Determine the charge density per unit length.

(b) Determine the current crossing the z-plane.

Solution:

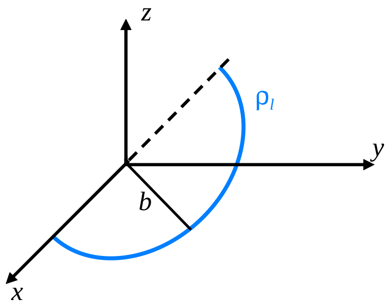
(a)

$$
\begin{aligned}
\rho_l &= \int \rho_v \, ds \\
&= \int_{r=0}^{a} \int_{\phi=0}^{2\pi} -cr^2 \cdot r \, dr \, d\phi = -2\pi c \left. \frac{r^4}{4} \right|_0^a = -\frac{\pi c a^4}{2} \quad \text{(C/m)}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{J} &= \rho_\text{v}\mathbf{u} = -\hat{\mathbf{z}}cr^2u \quad (\text{A/m}^2) \\
I &= \int \mathbf{J}\cdot d\mathbf{s} \\
&= \int_{r=0}^{a}\int_{\phi=0}^{2\pi} (-\hat{\mathbf{z}}cur^2)\cdot\hat{\mathbf{z}}r\,dr\,d\phi \\
&= -2\pi cu\int_0^a r^3\,dr = -\frac{\pi cua^4}{2} = \rho_l u. \quad (\text{A}).
\end{aligned}
$$

Problem 4.60 A line of charge of uniform density ρ_l occupies a semicircle of radius b as shown in the figure. Use the material presented in Example 4-4 to determine the electric field at the origin.



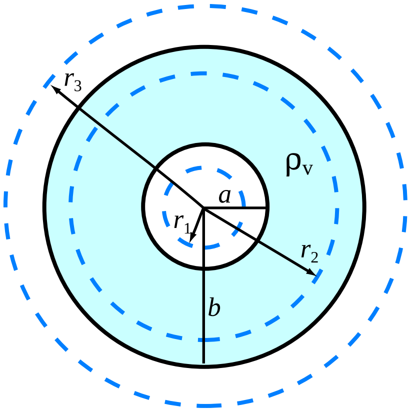
Solution: Since we have only half of a circle, we need to integrate the expression for $d\mathbf{E}_1$ given in Example 4-4 over ϕ from 0 to π. Before we do that, however, we need to set $h = 0$ (the problem asks for $\mathbf{E}$ at the origin). Hence,

$$
\begin{aligned}
d\mathbf{E}_1 &= \frac{\rho_l b}{4\pi\varepsilon_0}\,\frac{(-\hat{\mathbf{r}}b+\hat{\mathbf{z}}h)}{(b^2+h^2)^{3/2}}\,d\phi\Bigg|_{h=0} \\
&= \frac{-\hat{\mathbf{r}}\rho_l}{4\pi\varepsilon_0 b}\,d\phi \\
\mathbf{E}_1 &= \int_{\phi=0}^{\pi} d\mathbf{E}_1 = -\frac{-\hat{\mathbf{r}}\rho_l}{4\varepsilon_0 b}\,.
\end{aligned}
$$

Problem 4.61 A spherical shell with outer radius b surrounds a charge-free cavity of radius $a < b$. If the shell contains a charge density given by

$$
\rho_\text{v} = -\frac{\rho_{\text{v}0}}{R^2}\,, \qquad a \le R \le b,
$$

where $\rho_{\text{v}0}$ is a positive constant, determine $\mathbf{D}$ in all regions.



Solution: Symmetry dictates that **D** is radially oriented. Thus,

$$\mathbf{D} = \hat{\mathbf{R}} D_R.$$

At any R, Gauss's law gives

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

$$\int_S \hat{\mathbf{R}} D_R \cdot \hat{\mathbf{R}}\, ds = Q$$

$$4\pi R^2 D_R = Q$$

$$D_R = \frac{Q}{4\pi R^2}\,.$$

(a) For $R < a$, no charge is contained in the cavity. Hence, $Q = 0$, and

$$D_R = 0, \qquad R \leq a.$$

(b) For $a \leq R \leq b$,

$$\begin{aligned} Q = \int_{R=a}^{R} \rho_v \, dV &= \int_{R=a}^{R} -\frac{\rho_{v0}}{R^2} \cdot 4\pi R^2 \, dR \\ &= -4\pi\rho_{v0}(R-a). \end{aligned}$$

Hence,

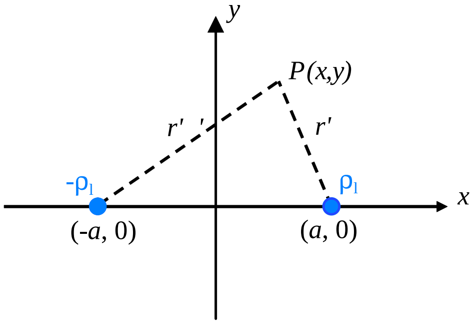
$$D_R = -\frac{\rho_{v0}(R-a)}{R^2}\,, \qquad a \leq R \leq b.$$

(c) For $R \geq b$,

$$Q = \int_{R=a}^{b} \rho_{\text{v}} \, dV = -4\pi\rho_{\text{v}0}(b-a)$$

$$D_R = -\frac{\rho_{\text{v}0}(b-a)}{R^2}, \qquad R \geq b.$$

Problem 4.62 Two infinite lines of charge, both parallel to the z-axis, lie in the x–z plane, one with density ρ_l and located at $x = a$ and the other with density $-\rho_l$ and located at $x = -a$. Obtain an expression for the electric potential $V(x,y)$ at a point $P(x,y)$ relative to the potential at the origin.



Solution: According to the result of Problem 4.30, the electric potential difference between a point at a distance r_1 and another at a distance r_2 from a line charge of density ρ_l is

$$V = \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{r_2}{r_1}\right).$$

Applying this result to the line charge at $x = a$, which is at a distance a from the origin:

$$\begin{aligned} V' &= \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{a}{r'}\right) \qquad (r_2 = a \text{ and } r_1 = r') \\ &= \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{a}{\sqrt{(x-a)^2+y^2}}\right). \end{aligned}$$

Similarly, for the negative line charge at $x = -a$,

$$\begin{aligned} V'' &= \frac{-\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{a}{r''}\right) \qquad (r_2 = a \text{ and } r_1 = r') \\ &= \frac{-\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{a}{\sqrt{(x+a)^2+y^2}}\right). \end{aligned}$$

The potential due to both lines is

$$V = V' + V'' = \frac{\rho_l}{2\pi\varepsilon_0}\left[\ln\left(\frac{a}{\sqrt{(x-a)^2+y^2}}\right) - \ln\left(\frac{a}{\sqrt{(x+a)^2+y^2}}\right)\right].$$

At the origin, $V = 0$, as it should be since the origin is the reference point. The potential is also zero along all points on the y-axis ($x = 0$).

Problem 4.63 A cylinder-shaped carbon resistor is 8 cm in length and its circular cross section has a diameter $d = 1$ mm.

- **(a)** Determine the resistance R.
- **(b)** To reduce its resistance by 40%, the carbon resistor is coated with a layer of copper of thickness t. Use the result of Problem 4.40 to determine t.

Solution:

(a) From (4.70), and using the value of σ for carbon from Appendix B,

$$R = \frac{l}{\sigma A} = \frac{l}{\sigma\pi(d/2)^2} = \frac{8\times10^{-2}}{3\times10^4\pi(10^{-3}/2)^2} = 3.4\ \Omega.$$

(b) The 40%-reduced resistance is:

$$R' = 0.6R = 0.6\times3.4 = 2.04\ \Omega.$$

Using the result of Problem 4.40:

$$R' = \frac{l}{\pi(\sigma_1 a^2 + \sigma_2(b^2-a^2))} = 2.04\ \Omega.$$

With $\sigma_1 = 3.4\times10^4$ S/m (carbon), $\sigma_2 = 5.8\times10^7$ S/m (copper), $a = 1\text{ mm}/2 = 5\times10^{-4}$ m, and b unknown, we have

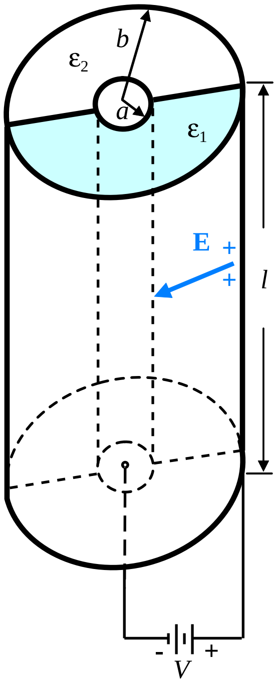
$$b = 5.00086\times10^{-4}\text{ m}$$

and

$$\begin{aligned} t = b - a &= (5.00086 - 5)\times10^{-4} \\ &= 0.00086\times10^{-4}\text{ m} = 0.086\ \mu\text{m}. \end{aligned}$$

Thus, the addition of a copper coating less than 0.1 μm in thickness reduces the resistance by 40%.

Problem 4.64 A coaxial capacitor consists of two concentric, conducting, cylindrical surfaces, one of radius a and another of radius b, as shown in the figure. The insulating layer separating the two conducting surfaces is divided equally into two semi-cylindrical sections, one filled with dielectric ε_1 and the other filled with dielectric ε_2.



(a) Develop an expression for C in terms of the length l and the given quantities.

(b) Evaluate the value of C for $a = 2$ mm, $b = 6$ mm, $\varepsilon_{r_1} = 2$, $\varepsilon_{r_2} = 4$, and $l = 4$ cm.

Solution:

(a) For the indicated voltage polarity, the **E** field inside the capacitor exists in only the dielectric materials and points radially inward. Let $\mathbf{E}_1$ be the field in dielectric ε_1 and $\mathbf{E}_2$ be the field in dielectric ε_2. At the interface between the two dielectric sections, $\mathbf{E}_1$ is parallel to $\mathbf{E}_2$ and both are tangential to the interface. Since boundary conditions require that the tangential components of $\mathbf{E}_1$ and $\mathbf{E}_2$ be the same, it follows that:

$$\mathbf{E}_1 = \mathbf{E}_2 = -\hat{\mathbf{r}}E.$$

At $r = a$ (surface of inner conductor), in medium 1, the boundary condition on **D**, as stated by (4.101), leads to

$$\mathbf{D}_1 = \varepsilon_1\mathbf{E}_1 = \hat{\mathbf{n}}\rho_{s1}$$
$$-\hat{\mathbf{r}}\varepsilon_1 E = \hat{\mathbf{r}}\rho_{s1}$$

or

$$\rho_{s1} = -\varepsilon_1 E.$$

Similarly, in medium 2

$$\rho_{s2} = -\varepsilon_2 E.$$

Thus, the **E** fields will be the same in the two dielectrics, but the charge densities will be different along the two sides of the inner conducting cylinder.

Since the same voltage applies for the two sections of the capacitor, we can treat them as two capacitors in parallel. For the capacitor half that includes dielectric ε_1, we can apply the results of Eqs. (4.114)–(4.116), but we have to keep in mind that Q is now the charge on only one half of the inner cylinder. Hence,

$$C_1 = \frac{\pi\varepsilon_1 l}{\ln(b/a)} .$$

Similarly,

$$C_2 = \frac{\pi\varepsilon_2 l}{\ln(b/a)} ,$$

and

$$C = C_1 + C_2 = \frac{\pi l(\varepsilon_1 + \varepsilon_2)}{\ln(b/a)} .$$

(b)

$$C = \frac{\pi \times 4 \times 10^{-2}(2+4) \times 8.85 \times 10^{-12}}{\ln(6/2)}$$
$$= 6.07 \text{ pF}.$$

Chapter 5: Magnetostatics

Lesson #30

Chapter — Section: 5-1

Topics: Magnetic forces and torques

Highlights:

- Lorentz force on a charged particle
- Magnetic force on a current in a magnetic field
- Torque on a loop

Special Illustrations:

- Examples 5-1

Lesson #31

Chapter — Section: 5-2

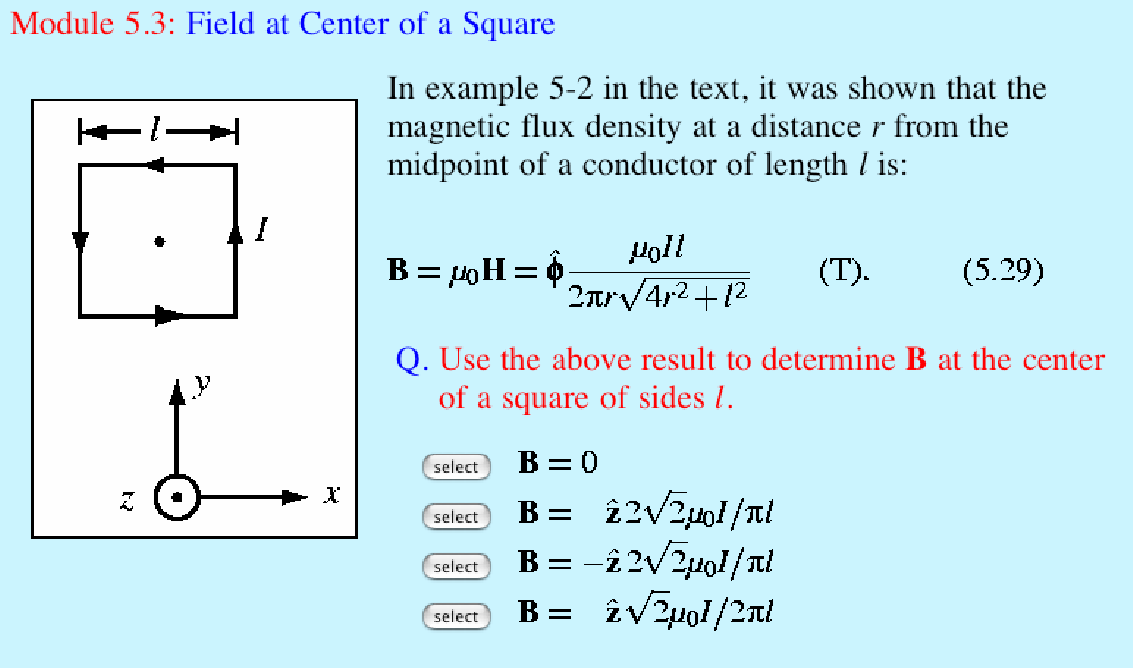
Topics: Biot-Savart law

Highlights:

- Magnetic field induction by electric currents
- Magnetic field due to linear conductor
- Magnetic dipole

Special Illustrations:

- Example 5-2
- Example 5-3
- CD-ROM Modules 5.3 and 5.4



Module 5.3: Field at Center of a Square

In example 5-2 in the text, it was shown that the magnetic flux density at a distance r from the midpoint of a conductor of length l is:

$$\mathbf{B} = \mu_0 \mathbf{H} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I l}{2\pi r \sqrt{4r^2 + l^2}} \quad \text{(T).} \qquad (5.29)$$

Q. Use the above result to determine **B** at the center of a square of sides l.

$\mathbf{B} = 0$

$\mathbf{B} = \hat{\mathbf{z}}\, 2\sqrt{2}\mu_0 I / \pi l$

$\mathbf{B} = -\hat{\mathbf{z}}\, 2\sqrt{2}\mu_0 I / \pi l$

$\mathbf{B} = \hat{\mathbf{z}} \sqrt{2}\mu_0 I / 2\pi l$

Lesson #32

Chapter — Section: 5-3, 5-4

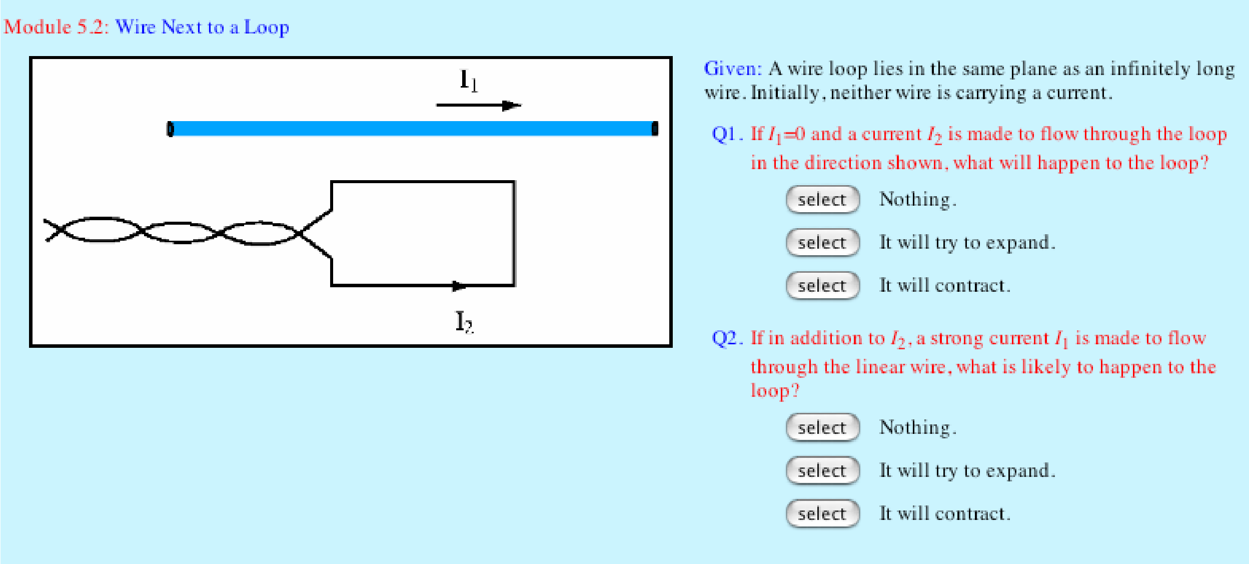
Topics: Magnetic force, Ampère's law

Highlights:

- Attraction and repulsion forces between currents
- Gauss's law for magnetics
- Ampère's law

Special Illustrations:

- Example 5-6
- CD-ROM Modules 5.1 and 5.2



Module 5.2: Wire Next to a Loop

Given: A wire loop lies in the same plane as an infinitely long wire. Initially, neither wire is carrying a current.

Q1. If $I_1=0$ and a current I_2 is made to flow through the loop in the direction shown, what will happen to the loop?

- Nothing.
- It will try to expand.
- It will contract.

Q2. If in addition to I_2, a strong current I_1 is made to flow through the linear wire, what is likely to happen to the loop?

- Nothing.
- It will try to expand.
- It will contract.

Lesson #33

Chapter — Section: 5-5, 5-6

Topics: Vector magnetic potential, magnetic materials

Highlights:

- Relation of **A** to **B**
- Vector Poisson's Eq.
- Magnetic permeability
- Ferromagnetism, hysteresis

Special Illustrations:

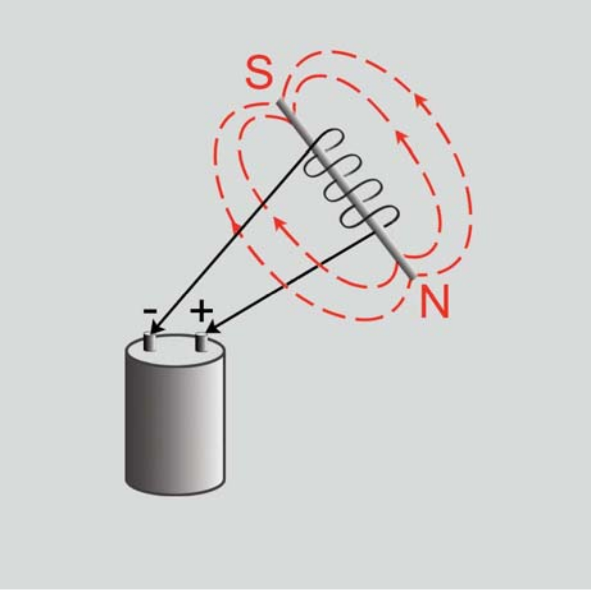
- Technology Brief on "Electromagnetic and magnetic switches" (CD-ROM)

Electromagnets and Magnetic Relays

William Sturgeon developed the first practical electromagnet in the 1820s. Today the principle of the electromagnet is used in motors, relay switches in read/write heads for hard disks and tape drives, loudspeakers, magnetic levitation and many other applications.

Basic Principle

Electromagnets can be constructed in various shapes, including the linear solenoid described in Section 5-8.1. When an electric current generated by a power source, such as a battery, flows through the wire coiled around the central core, it induces a magnetic field with field lines resembling those generated by a bar magnet (A1). The strength of the magnetic field is proportional to the current, the number of turns, and the magnetic permeability of the core material. By using a ferromagnetic core, the field strength can be increased by several orders of magnitude, depending on the purity of the iron material. When subjected to a magnetic field, ferromagnetic materials, such as iron or nickel, get magnetized and act like magnets themselves.



Lesson #34

Chapter — Section: 5-7

Topics: Boundary conditions

Highlights:

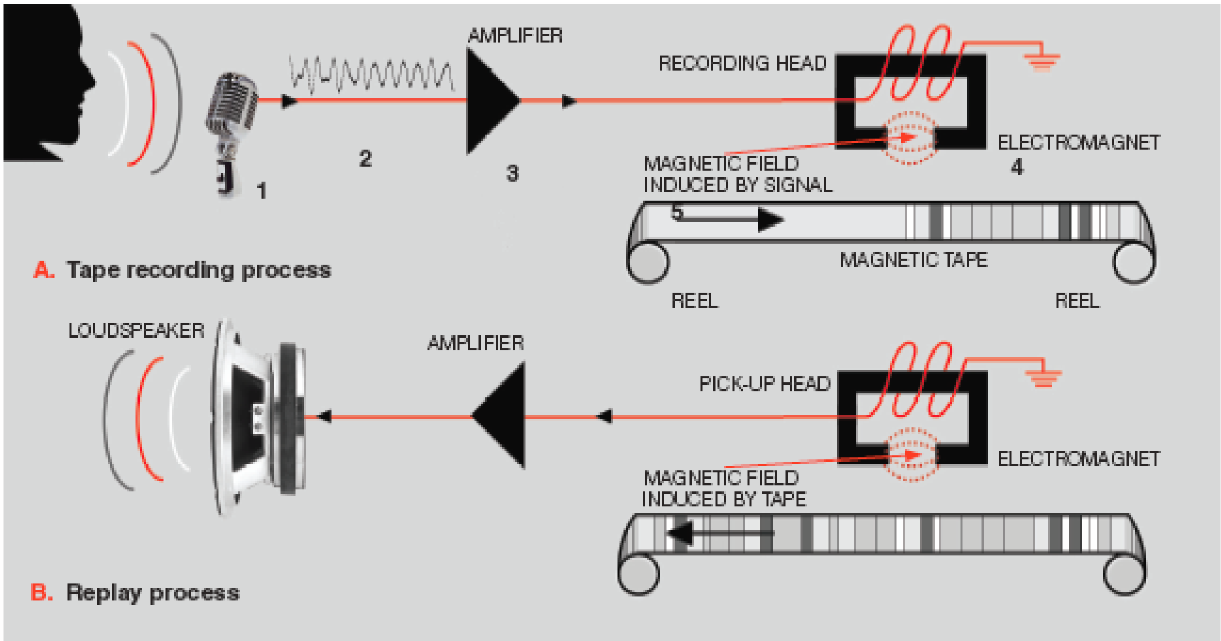
- Analogy with electric-field boundary conditions

Special Illustrations:

- Technology Brief on "Magnetic Recording" (CD-ROM)

Magnetic Recording

Valdemar Poulsen, a Danish engineer, invented magnetic recording by demonstrating in 1900 that speech can be recorded on a thin steel wire using a simple electromagnet. Magnetic tapes were developed as an alternative medium to wires in the 1940s and became very popular for recording and playing music well into the 1960s. Videotapes were introduced in the late 1950s for recording motion pictures for later replay on television. Because video signals occupy a much wider bandwidth, tape speeds for video recording (past the magnetic head) have to be at rates on the order of 5 m/s, compared with only 0.3 m/s for audio. Other types of magnetic recording media were developed since then, including the flexible plastic disks called "floppies," the hard disks made of glass or aluminum, the magnetic drum, and the magnetic bubble memory. All take advantage of the same fundamental principle of being able to store electrical information through selective magnetization of a magnetic material, as well as the ability to retrieve it (playback) when so desired.



Lesson #35

Chapter — Section: 5-8

Topics: Inductance

Highlights:

- Solenoid
- Self inductance

Special Illustrations:

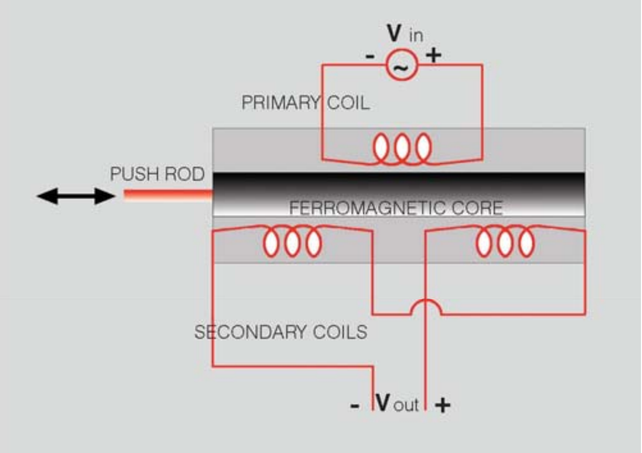
- Example 5-8
- Technology Brief on "Inductive Sensors" (CD-ROM)

Inductive Sensors

Magnetic coupling between different coils forms the basis of several different types of inductive sensors. Applications include the measurement of position and displacement (with sub-millimeter resolution) in device fabrications processes, proximity detection of conductive objects, and other related applications.

Linear Variable Differential Transformer (LVDT)

A LVDT comprises a primary coil connected to an ac source, typically a sine wave at a frequency in the 1–10 KHz range, and a pair of secondary coils, all sharing a common ferromagnetic core (A1). The magnetic core serves to couple the magnetic flux generated by the primary coil into the two secondaries, thereby inducing an output voltage across each of them. The secondary coils are connected in opposition, so that when the core is positioned at the magnetic center of the LVDT, the individual output signals of the secondaries cancel each other out, producing a null output voltage. The core is connected to the outside world via a nonmagnetic rod. When the rod moves the core away from the magnetic center, the magnetic fluxes induced in the secondary coils are no longer equal, resulting in a non-zero output voltage. The LVDT is called a "linear" transformer because the output voltage is a linear function of displacement over a wide operating range.



A1. LVDT circuit

Lesson #36

Chapter — Section: 5-9

Topics: Magnetic energy

Highlights:

- Magnetic energy density
- Magnetic energy in a coax

Special Illustrations:

- Example 5-9

Chapter 5

Sections 5-1: Forces and Torques

Problem 5.1 An electron with a speed of 8×10^6 m/s is projected along the positive x-direction into a medium containing a uniform magnetic flux density $\mathbf{B} = (\hat{\mathbf{x}}4 - \hat{\mathbf{z}}3)$ T. Given that $e = 1.6 \times 10^{-19}$ C and the mass of an electron is $m_e = 9.1 \times 10^{-31}$ kg, determine the initial acceleration vector of the electron (at the moment it is projected into the medium).

Solution: The acceleration vector of a free particle is the net force vector divided by the particle mass. Neglecting gravity, and using Eq. (5.3), we have

$$\begin{aligned}\mathbf{a} = \frac{\mathbf{F}_\text{m}}{m_\text{e}} = \frac{-e}{m_\text{e}}\mathbf{u} \times \mathbf{B} &= \frac{-1.6 \times 10^{-19}}{9.1 \times 10^{-31}}(\hat{\mathbf{x}}8 \times 10^6) \times (\hat{\mathbf{x}}4 - \hat{\mathbf{z}}3) \\ &= -\hat{\mathbf{y}}4.22 \times 10^{18} \quad (\text{m/s}^2).\end{aligned}$$

Problem 5.2 When a particle with charge q and mass m is introduced into a medium with a uniform field $\mathbf{B}$ such that the initial velocity of the particle $\mathbf{u}$ is perpendicular to $\mathbf{B}$, as shown in Fig. 5-31 (P5.2), the magnetic force exerted on the particle causes it to move in a circle of radius a. By equating $\mathbf{F}_\text{m}$ to the centripetal force on the particle, determine a in terms of q, m, u, and $\mathbf{B}$.

Solution: The centripetal force acting on the particle is given by $F_\text{c} = mu^2/a$.

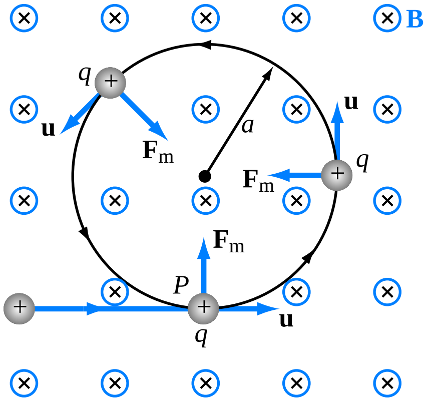


Figure P5.2: Particle of charge q projected with velocity $\mathbf{u}$ into a medium with a uniform field $\mathbf{B}$ perpendicular to $\mathbf{u}$ (Problem 5.2).

Equating F_c to F_m given by Eq. (5.4), we have $mu^2/a = quB\sin\theta$. Since the magnetic field is perpendicular to the particle velocity, $\sin\theta = 1$. Hence, $a = mu/qB$.

Problem 5.3 The circuit shown in Fig. 5-32 (P5.3) uses two identical springs to support a 10-cm-long horizontal wire with a mass of 20 g. In the absence of a magnetic field, the weight of the wire causes the springs to stretch a distance of 0.2 cm each. When a uniform magnetic field is turned on in the region containing the horizontal wire, the springs are observed to stretch an additional 0.5 cm. What is the intensity of the magnetic flux density **B**? The force equation for a spring is $F = kd$, where k is the spring constant and d is the distance it has been stretched.

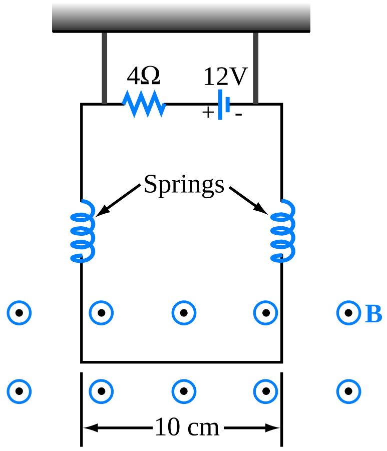


Figure P5.3: Configuration of Problem 5.3.

Solution: Springs are characterized by a spring constant k where $F = kd$ is the force exerted on the spring and d is the amount the spring is stretched from its rest configuration. In this instance, each spring sees half the weight of the wire:

$$F = \tfrac{1}{2}mg = kd, \qquad k = \frac{mg}{2d} = \frac{20 \times 10^{-3} \times 9.8}{2 \times 2 \times 10^{-3}} = 49 \quad \text{(N/m)}.$$

Therefore, when the springs are further stretched by an additional 0.5 cm, this amounts to an additional force of $F = 49\ \text{N/m} \times (5 \times 10^{-3}\ \text{m}) = 245$ mN per spring, or a total additional force of $F = 0.49$ N. This force is equal to the force exerted on the wire by the interaction of the magnetic field and the current as described by Eq. (5.12): $\mathbf{F}_\text{m} = I\boldsymbol{\ell} \times \mathbf{B}$, where $\boldsymbol{\ell}$ and $\mathbf{B}$ are at right angles. Moreover $\boldsymbol{\ell} \times \mathbf{B}$ is in the downward direction, and $I = V/R = 12\text{ V}/4\ \Omega = 3$ A. Therefore,

$$|\mathbf{F}_\text{m}| = |I||\boldsymbol{\ell}||\mathbf{B}|, \qquad |\mathbf{B}| = \frac{|\mathbf{F}_\text{m}|}{|I||\boldsymbol{\ell}|} = \frac{0.49}{3 \times 0.1} = 1.63 \quad \text{(T)}.$$

Problem 5.4 The rectangular loop shown in Fig. 5-33 (P5.4) consists of 20 closely wrapped turns and is hinged along the z-axis. The plane of the loop makes an angle of 30° with the y-axis, and the current in the windings is 0.5 A. What is the magnitude of the torque exerted on the loop in the presence of a uniform field $\mathbf{B} = \hat{\mathbf{y}}2.4$ T? When viewed from above, is the expected direction of rotation clockwise or counterclockwise?

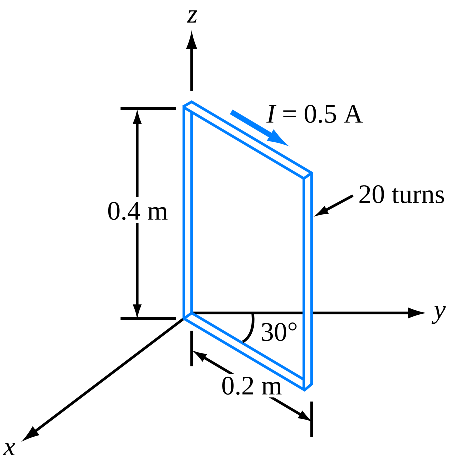


Figure P5.4: Hinged rectangular loop of Problem 5.4.

Solution: The magnetic torque on a loop is given by $\mathbf{T} = \mathbf{m} \times \mathbf{B}$ (Eq. (5.20)), where $\mathbf{m} = \hat{\mathbf{n}}NIA$ (Eq. (5.19)). For this problem, it is given that $I = 0.5$ A, $N = 20$ turns, and $A = 0.2\text{ m} \times 0.4\text{ m} = 0.08\text{ m}^2$. From the figure, $\hat{\mathbf{n}} = -\hat{\mathbf{x}}\cos 30° + \hat{\mathbf{y}}\sin 30°$. Therefore, $\mathbf{m} = \hat{\mathbf{n}}0.8\ (\text{A}\cdot\text{m}^2)$ and $\mathbf{T} = \hat{\mathbf{n}}0.8\ (\text{A}\cdot\text{m}^2) \times \hat{\mathbf{y}}2.4\text{ T} = -\hat{\mathbf{z}}1.66$ (N·m). As the torque is negative, the direction of rotation is clockwise, looking from above.

Problem 5.5 In a cylindrical coordinate system, a 2-m-long straight wire carrying a current of 5 A in the positive z-direction is located at $r = 4$ cm, $\phi = \pi/2$, and $-1\text{ m} \le z \le 1$ m.

(a) If $\mathbf{B} = \hat{\mathbf{r}}0.2\cos\phi$ (T), what is the magnetic force acting on the wire?

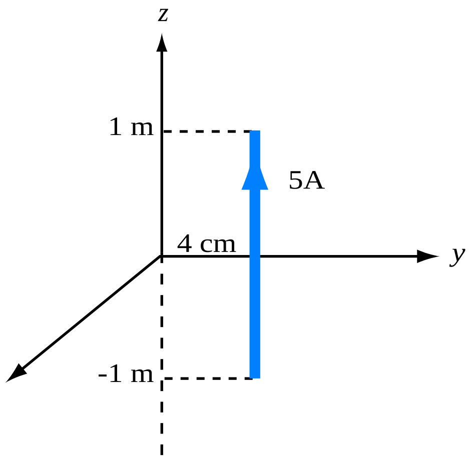


Figure P5.5: Problem 5.5.

(b) How much work is required to rotate the wire once about the *z*-axis in the negative ϕ-direction (while maintaining $r = 4$ cm)?

(c) At what angle ϕ is the force a maximum?

Solution:

(a)

$$\begin{aligned}\mathbf{F} &= I\boldsymbol{\ell}\times\mathbf{B}\\ &= 5\hat{\mathbf{z}}2\times[\hat{\mathbf{r}}0.2\cos\phi]\\ &= \hat{\boldsymbol{\phi}}2\cos\phi.\end{aligned}$$

At $\phi = \pi/2$, $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}$. Hence,

$$\mathbf{F} = -\hat{\mathbf{x}}2\cos(\pi/2) = 0.$$

(b)

$$\begin{aligned}W &= \int_{\phi=0}^{2\pi}\mathbf{F}\cdot d\mathbf{l} = \int_0^{2\pi}\hat{\boldsymbol{\phi}}[2\cos\phi]\cdot(-\hat{\boldsymbol{\phi}})r\,d\phi\Big|_{r=4\text{ cm}}\\ &= -2r\int_0^{2\pi}\cos\phi\,d\phi\Big|_{r=4\text{ cm}} = -8\times10^{-2}[\sin\phi]_0^{2\pi} = 0.\end{aligned}$$

The force is in the $+\hat{\boldsymbol{\phi}}$-direction, which means that rotating it in the $-\hat{\boldsymbol{\phi}}$-direction would require work. However, the force varies as $\cos\phi$, which means it is positive when $-\pi/2 \le \phi \le \pi/2$ and negative over the second half of the circle. Thus, work is provided by the force between $\phi = \pi/2$ and $\phi = -\pi/2$ (when rotated in the $-\hat{\boldsymbol{\phi}}$-direction), and work is supplied for the second half of the rotation, resulting in a net work of zero.

(c) The force $\mathbf{F}$ is maximum when $\cos\phi = 1$, or $\phi = 0$.

Problem 5.6 A 20-turn rectangular coil with side $l = 20$ cm and $w = 10$ cm is placed in the y–z plane as shown in Fig. 5-34 (P5.6).

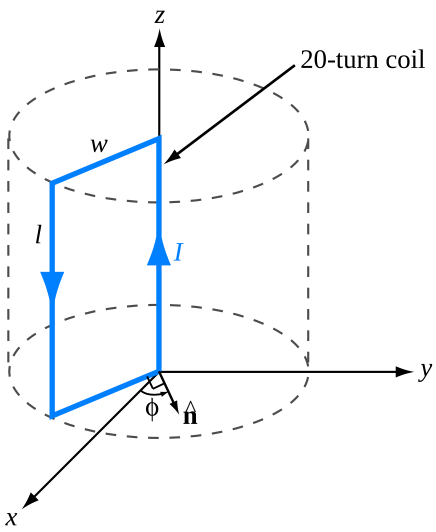


Figure P5.6: Rectangular loop of Problem 5.6.

(a) If the coil, which carries a current $I = 10$ A, is in the presence of a magnetic flux density

$$\mathbf{B} = 2 \times 10^{-2}(\hat{\mathbf{x}} + \hat{\mathbf{y}}2) \quad \text{(T)},$$

determine the torque acting on the coil.

(b) At what angle ϕ is the torque zero?

(c) At what angle ϕ is the torque maximum? Determine its value.

Solution:

(a) The magnetic field is in direction $(\hat{\mathbf{x}} + \hat{\mathbf{y}}2)$, which makes an angle $\phi_0 = \tan^{-1}\frac{2}{1} = 63.43°$.

The magnetic moment of the loop is

$$\mathbf{m} = \hat{\mathbf{n}}NIA = \hat{\mathbf{n}}20 \times 10 \times (30 \times 10) \times 10^{-4} = \hat{\mathbf{n}}6 \quad (\text{A}\cdot\text{m}^2),$$

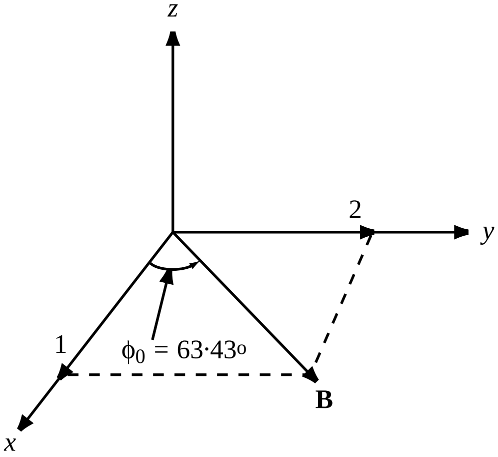


Figure P5.6: (a) Direction of **B**.

where $\hat{\mathbf{n}}$ is the surface normal in accordance with the right-hand rule. When the loop is in the negative-*y* of the *y*–*z* plane, $\hat{\mathbf{n}}$ is equal to $\hat{\mathbf{x}}$, but when the plane of the loop is moved to an angle ϕ, $\hat{\mathbf{n}}$ becomes

$$\begin{aligned}
\hat{\mathbf{n}} &= \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi, \\
\mathbf{T} = \mathbf{m}\times\mathbf{B} &= \hat{\mathbf{n}}6 \times 2\times 10^{-2}(\hat{\mathbf{x}}+\hat{\mathbf{y}}2) \\
&= (\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi)\,6 \times 2\times 10^{-2}(\hat{\mathbf{x}}+\hat{\mathbf{y}}2) \\
&= \hat{\mathbf{z}}0.12[2\cos\phi - \sin\phi] \quad (\text{N}\cdot\text{m}).
\end{aligned}$$

(b) The torque is zero when

$$2\cos\phi - \sin\phi = 0,$$

or

$$\tan\phi = 2, \qquad \phi = 63.43^\circ \text{ or } -116.57^\circ.$$

Thus, when $\hat{\mathbf{n}}$ is parallel to $\mathbf{B}$, $\mathbf{T} = 0$.

(c) The torque is a maximum when $\hat{\mathbf{n}}$ is perpendicular to $\mathbf{B}$, which occurs at

$$\phi = 63.43 \pm 90^\circ = -26.57^\circ \text{ or } +153.43^\circ.$$

Mathematically, we can obtain the same result by taking the derivative of **T** and equating it to zero to find the values of ϕ at which $|\mathbf{T}|$ is a maximum. Thus,

$$\frac{\partial T}{\partial\phi} = \frac{\partial}{\partial\phi}(0.12(2\cos\phi - \sin\phi)) = 0$$

or

$$-2\sin\phi + \cos\phi = 0,$$

which gives $\tan\phi = -\frac{1}{2}$, or

$$\phi = -26.57^\circ \text{ or } 153.43^\circ,$$

at which $\mathbf{T} = \hat{\mathbf{z}}0.27$ (N·m).

Section 5-2: Biot–Savart Law

Problem 5.7 An 8 cm × 12 cm rectangular loop of wire is situated in the x–y plane with the center of the loop at the origin and its long sides parallel to the x-axis. The loop has a current of 50 A flowing with clockwise direction (when viewed from above). Determine the magnetic field at the center of the loop.

Solution: The total magnetic field is the vector sum of the individual fields of each of the four wire segments: $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4$. An expression for the magnetic field from a wire segment is given by Eq. (5.29).

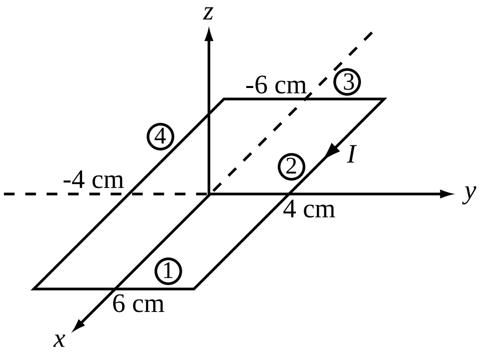


Figure P5.7: Problem 5.7.

For all segments shown in Fig. P5.7, the combination of the direction of the current and the right-hand rule gives the direction of the magnetic field as $-z$ direction at the origin. With $r = 6$ cm and $l = 8$ cm,

$$\begin{aligned}\mathbf{B}_1 &= -\hat{\mathbf{z}}\frac{\mu I l}{2\pi r\sqrt{4r^2 + l^2}} \\ &= -\hat{\mathbf{z}}\frac{4\pi \times 10^{-7} \times 50 \times 0.08}{2\pi \times 0.06 \times \sqrt{4 \times 0.06^2 + 0.08^2}} = -\hat{\mathbf{z}}9.24 \times 10^{-5} \quad \text{(T)}.\end{aligned}$$

For segment 2, $r = 4$ cm and $l = 12$ cm,

$$\begin{aligned}\mathbf{B}_2 &= -\hat{\mathbf{z}}\frac{\mu I l}{2\pi r\sqrt{4r^2+l^2}} \\ &= -\hat{\mathbf{z}}\frac{4\pi\times10^{-7}\times50\times0.12}{2\pi\times0.04\times\sqrt{4\times0.04^2+0.12^2}} = -\hat{\mathbf{z}}20.80\times10^{-5} \quad \text{(T)}.\end{aligned}$$

Similarly,

$$\mathbf{B}_3 = -\hat{\mathbf{z}}9.24\times10^{-5} \quad \text{(T)}, \qquad \mathbf{B}_4 = -\hat{\mathbf{z}}20.80\times10^{-5} \quad \text{(T)}.$$

The total field is then $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4 = -\hat{\mathbf{z}}0.60$ (mT).

Problem 5.8 Use the approach outlined in Example 5-2 to develop an expression for the magnetic field **H** at an arbitrary point P due to the linear conductor defined by the geometry shown in Fig. 5-35 (P5.8). If the conductor extends between $z_1 = 3$ m and $z_2 = 7$ m and carries a current $I = 15$ A, find **H** at $P(2,\phi,0)$.

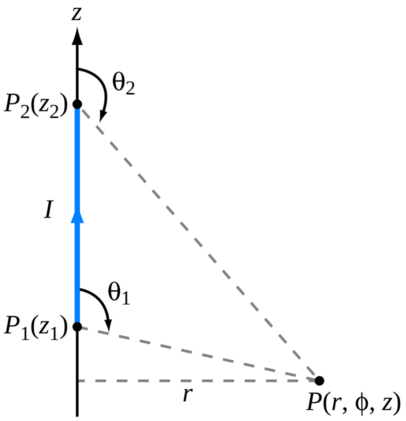


Figure P5.8: Current-carrying linear conductor of Problem 5.8.

Solution: The solution follows Example 5-2 up through Eq. (5.27), but the expressions for the cosines of the angles should be generalized to read as

$$\cos\theta_1 = \frac{z-z_1}{\sqrt{r^2+(z-z_1)^2}}, \qquad \cos\theta_2 = \frac{z-z_2}{\sqrt{r^2+(z-z_2)^2}}$$

instead of the expressions in Eq. (5.28), which are specialized to a wire centered at the origin. Plugging these expressions back into Eq. (5.27), the magnetic field is given as

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi r} \left(\frac{z - z_1}{\sqrt{r^2 + (z - z_1)^2}} - \frac{z - z_2}{\sqrt{r^2 + (z - z_2)^2}} \right).$$

For the specific geometry of Fig. P5.8,

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{15}{4\pi \times 2} \left[\frac{0-3}{\sqrt{3^2+2^2}} - \frac{0-7}{\sqrt{7^2+2^2}} \right] = \hat{\boldsymbol{\phi}} 77.4 \times 10^{-3} \text{ (A/m)} = \hat{\boldsymbol{\phi}} 77.4 \quad \text{(mA/m)}.$$

Problem 5.9 The loop shown in Fig. 5-36 (P5.9) consists of radial lines and segments of circles whose centers are at point P. Determine the magnetic field $\mathbf{H}$ at P in terms of a, b, θ, and I.

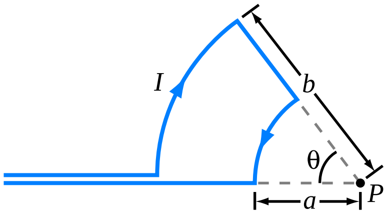


Figure P5.9: Configuration of Problem 5.9.

Solution: From the solution to Example 5-3, if we denote the z-axis as passing out of the page through point P, the magnetic field pointing out of the page at P due to the current flowing in the outer arc is $\mathbf{H}_{\text{outer}} = -\hat{\mathbf{z}} I\theta / 4\pi b$ and the field pointing out of the page at P due to the current flowing in the inner arc is $\mathbf{H}_{\text{inner}} = \hat{\mathbf{z}} I\theta / 4\pi a$. The other wire segments do not contribute to the magnetic field at P. Therefore, the total field flowing directly out of the page at P is

$$\mathbf{H} = \mathbf{H}_{\text{outer}} + \mathbf{H}_{\text{inner}} = \hat{\mathbf{z}} \frac{I\theta}{4\pi} \left(\frac{1}{a} - \frac{1}{b} \right) = \hat{\mathbf{z}} \frac{I\theta (b-a)}{4\pi ab}.$$

Problem 5.10 An infinitely long, thin conducting sheet defined over the space $0 \le x \le w$ and $-\infty \le y \le \infty$ is carrying a current with a uniform surface current

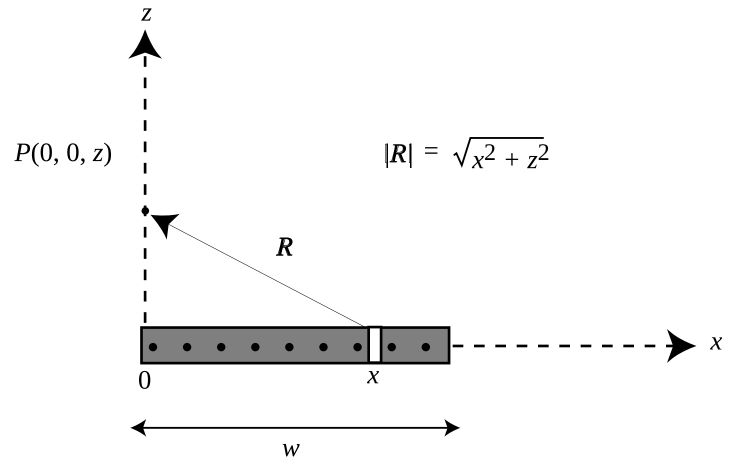


Figure P5.10: Conducting sheet of width w in x–y plane.

density $\mathbf{J}_s = \hat{\mathbf{y}}5$ (A/m). Obtain an expression for the magnetic field at point $P(0,0,z)$ in Cartesian coordinates.

Solution: The sheet can be considered to be a large number of infinitely long but narrow wires each dx wide lying next to each other, with each carrying a current $I_x = J_s\,dx$. The wire at a distance x from the origin is at a distance vector $\mathbf{R}$ from point P, with

$$\mathbf{R} = -\hat{\mathbf{x}}x + \hat{\mathbf{z}}z.$$

Equation (5.30) provides an expression for the magnetic field due to an infinitely long wire carrying a current I as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\hat{\boldsymbol{\phi}}I}{2\pi r}\,.$$

We now need to adapt this expression to the present situation by replacing I with $I_x = J_s\,dx$, replacing r with $R = (x^2+z^2)^{1/2}$, as shown in Fig. P5.10, and by assigning the proper direction for the magnetic field. From the Biot–Savart law, the direction of $\mathbf{H}$ is governed by $\mathbf{l}\times\mathbf{R}$, where $\mathbf{l}$ is the direction of current flow. In the present case, $\mathbf{l}$ is in the $\hat{\mathbf{y}}$ direction. Hence, the direction of the field is

$$\frac{\mathbf{l}\times\mathbf{R}}{|\mathbf{l}\times\mathbf{R}|} = \frac{\hat{\mathbf{y}}\times(-\hat{\mathbf{x}}x+\hat{\mathbf{z}}z)}{|\hat{\mathbf{y}}\times(-\hat{\mathbf{x}}x+\hat{\mathbf{z}}z)|} = \frac{\hat{\mathbf{x}}z+\hat{\mathbf{z}}x}{(x^2+z^2)^{1/2}}\,.$$

Therefore, the field $d\mathbf{H}$ due to the current I_x is

$$d\mathbf{H} = \frac{\hat{\mathbf{x}}z + \hat{\mathbf{z}}x}{(x^2+z^2)^{1/2}}\,\frac{I_x}{2\pi R} = \frac{(\hat{\mathbf{x}}z + \hat{\mathbf{z}}x)J_s\,dx}{2\pi(x^2+z^2)}\,,$$

and the total field is

$$\begin{aligned}
\mathbf{H}(0,0,z) &= \int_{x=0}^{w} (\hat{\mathbf{x}}z + \hat{\mathbf{z}}x)\,\frac{J_s\,dx}{2\pi(x^2+z^2)}\\
&= \frac{J_s}{2\pi}\int_{x=0}^{w} (\hat{\mathbf{x}}z + \hat{\mathbf{z}}x)\,\frac{dx}{x^2+z^2}\\
&= \frac{J_s}{2\pi}\left(\hat{\mathbf{x}}z\int_{x=0}^{w}\frac{dx}{x^2+z^2} + \hat{\mathbf{z}}\int_{x=0}^{w}\frac{x\,dx}{x^2+z^2}\right)\\
&= \frac{J_s}{2\pi}\left(\hat{\mathbf{x}}z\left(\frac{1}{z}\tan^{-1}\left(\frac{x}{z}\right)\right)\bigg|_{x=0}^{w} + \hat{\mathbf{z}}\left(\tfrac{1}{2}\ln(x^2+z^2)\right)\Big|_{x=0}^{w}\right)\\
&= \frac{5}{2\pi}\left[\hat{\mathbf{x}}2\pi\tan^{-1}\left(\frac{w}{z}\right) + \hat{\mathbf{z}}\tfrac{1}{2}(\ln(w^2+z^2) - \ln(0+z^2))\right] \quad \text{for } z \neq 0,\\
&= \frac{5}{2\pi}\left[\hat{\mathbf{x}}2\pi\tan^{-1}\left(\frac{w}{z}\right) + \hat{\mathbf{z}}\tfrac{1}{2}\ln\left(\frac{w^2+z^2}{z^2}\right)\right] \quad \text{(A/m)} \quad \text{for } z \neq 0.
\end{aligned}$$

An alternative approach is to employ Eq. (5.24a) directly.

Problem 5.11 An infinitely long wire carrying a 25-A current in the positive x-direction is placed along the x-axis in the vicinity of a 20-turn circular loop located in the x–y plane as shown in Fig. 5-37 (P5.11(a)). If the magnetic field at the center of the loop is zero, what is the direction and magnitude of the current flowing in the loop?

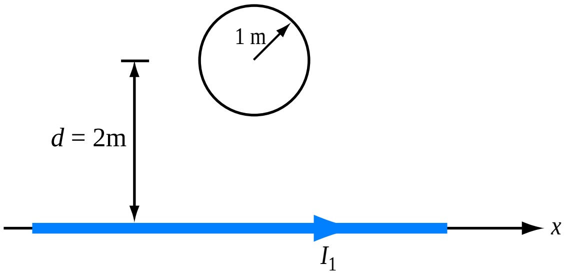


Figure P5.11: (a) Circular loop next to a linear current (Problem 5.11).

Solution: From Eq. (5.30), the magnetic flux density at the center of the loop due to

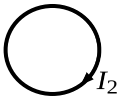


Figure P5.11: (b) Direction of I_2.

the wire is

$$\mathbf{B}_1 = \hat{\mathbf{z}}\,\frac{\mu_0}{2\pi d}\,I_1$$

where $\hat{\mathbf{z}}$ is out of the page. Since the net field is zero at the center of the loop, I_2 must be clockwise, as seen from above, in order to oppose I_1. The field due to I_2 is, from Eq. (5.35),

$$\mathbf{B} = \mu_0\mathbf{H} = -\hat{\mathbf{z}}\,\frac{\mu_0 N I_2}{2a}\,.$$

Equating the magnitudes of the two fields, we obtain the result

$$\frac{NI_2}{2a} = \frac{I_1}{2\pi d}\,,$$

or

$$I_2 = \frac{2aI_1}{2\pi N d} = \frac{1\times 25}{\pi\times 20\times 2} = 0.2\ \text{A}.$$

Problem 5.12 Two infinitely long, parallel wires carry 6-A currents in opposite directions. Determine the magnetic flux density at point P in Fig. 5-38 (P5.12).

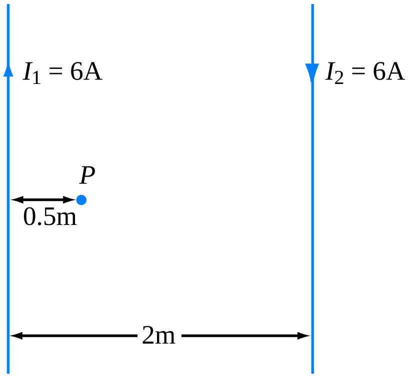


Figure P5.12: Arrangement for Problem 5.12.

Solution:

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I_1}{2\pi(0.5)} + \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I_2}{2\pi(1.5)} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{\pi}\,(6+2) = \hat{\boldsymbol{\phi}}\,\frac{8\mu_0}{\pi} \quad \text{(T)}.$$

Problem 5.13 A long, East-West oriented power cable carrying an unknown current I is at a height of 8 m above the Earth's surface. If the magnetic flux density recorded by a magnetic-field meter placed at the surface is 15 μT when the current is flowing through the cable and 20 μT when the current is zero, what is the magnitude of I?

Solution: The power cable is producing a magnetic flux density that opposes Earth's, own magnetic field. An East–West cable would produce a field whose direction at the surface is along North–South. The flux density due to the cable is

$$B = (20 - 15)\ \mu\text{T} = 5\mu\text{T}.$$

As a magnet, the Earth's field lines are directed from the South Pole to the North Pole inside the Earth and the opposite on the surface. Thus the lines at the surface are from North to South, which means that the field created by the cable is from South to North. Hence, by the right-hand rule, the current direction is toward the East. Its magnitude is obtained from

$$5\ \mu\text{T} = 5 \times 10^{-6} = \frac{\mu_0 I}{2\pi d} = \frac{4\pi \times 10^{-7} I}{2\pi \times 8},$$

which gives $I = 200$ A.

Problem 5.14 Two parallel, circular loops carrying a current of 40 A each are arranged as shown in Fig. 5-39 (P5.14). The first loop is situated in the x–y plane with its center at the origin and the second loop's center is at $z = 2$ m. If the two loops have the same radius $a = 3$ m, determine the magnetic field at:

(a) $z = 0$,
(b) $z = 1$ m,
(c) $z = 2$ m.

Solution: The magnetic field due to a circular loop is given by (5.34) for a loop in the x–y plane carrying a current I in the $+\hat{\boldsymbol{\phi}}$-direction. Considering that the bottom loop in Fig. P5.14 is in the x–y plane, but the current direction is along $-\hat{\boldsymbol{\phi}}$,

$$\mathbf{H}_1 = -\hat{\mathbf{z}}\,\frac{Ia^2}{2(a^2 + z^2)^{3/2}},$$

where z is the observation point along the z-axis. For the second loop, which is at a height of 2 m, we can use the same expression but z should be replaced with $(z - 2)$. Hence,

$$\mathbf{H}_2 = -\hat{\mathbf{z}}\,\frac{Ia^2}{2[a^2 + (z-2)^2]^{3/2}}.$$

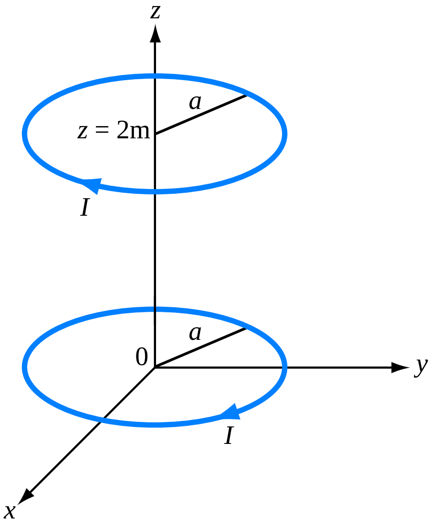


Figure P5.14: Parallel circular loops of Problem 5.14.

The total field is

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = -\hat{\mathbf{z}}\frac{Ia^2}{2}\left[\frac{1}{(a^2+z^2)^{3/2}} + \frac{1}{[a^2+(z-2)^2]^{3/2}}\right] \text{ A/m.}$$

(a) At $z = 0$, and with $a = 3$ m and $I = 40$ A,

$$\mathbf{H} = -\hat{\mathbf{z}}\frac{40\times 9}{2}\left[\frac{1}{3^3} + \frac{1}{(9+4)^{3/2}}\right] = -\hat{\mathbf{z}}10.5 \text{ A/m.}$$

(b) At $z = 1$ m (midway between the loops):

$$\mathbf{H} = -\hat{\mathbf{z}}\frac{40\times 9}{2}\left[\frac{1}{(9+1)^{3/2}} + \frac{1}{(9+1)^{3/2}}\right] = -\hat{\mathbf{z}}11.38 \text{ A/m.}$$

(c) At $z = 2$ m, **H** should be the same as at $z = 0$. Thus,

$$\mathbf{H} = -\hat{\mathbf{z}}10.5 \text{ A/m.}$$

Section 5-3: Forces between Currents

Problem 5.15 The long, straight conductor shown in Fig. 5-40 (P5.15) lies in the plane of the rectangular loop at a distance $d = 0.1$ m. The loop has dimensions

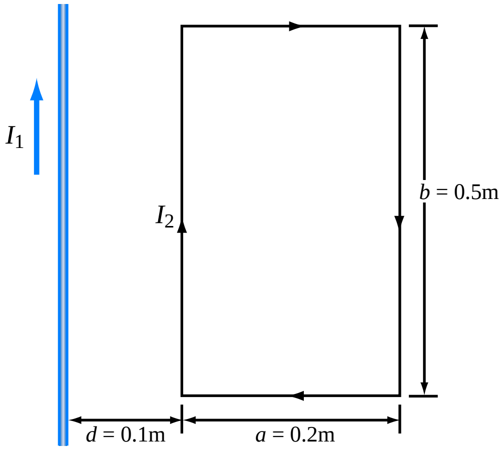


Figure P5.15: Current loop next to a conducting wire (Problem 5.15).

$a = 0.2$ m and $b = 0.5$ m, and the currents are $I_1 = 20$ A and $I_2 = 30$ A. Determine the net magnetic force acting on the loop.

Solution: The net magnetic force on the loop is due to the magnetic field surrounding the wire carrying current I_1. The magnetic forces on the loop as a whole due to the current in the loop itself are canceled out by symmetry. Consider the wire carrying I_1 to coincide with the z-axis, and the loop to lie in the $+x$ side of the x-z plane. Assuming the wire and the loop are surrounded by free space or other nonmagnetic material, Eq. (5.30) gives

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I_1}{2\pi r}.$$

In the plane of the loop, this magnetic field is

$$\mathbf{B} = \hat{\mathbf{y}} \frac{\mu_0 I_1}{2\pi x}.$$

Then, from Eq. (5.12), the force on the side of the loop nearest the wire is

$$\mathbf{F}_{\text{m1}} = I_2 \boldsymbol{\ell} \times \mathbf{B} = I_2(\hat{\mathbf{z}}b) \times \left(\hat{\mathbf{y}} \frac{\mu_0 I_1}{2\pi x} \right) \bigg|_{x=d} = -\hat{\mathbf{x}} \frac{\mu_0 I_1 I_2 b}{2\pi d}.$$

The force on the side of the loop farthest from the wire is

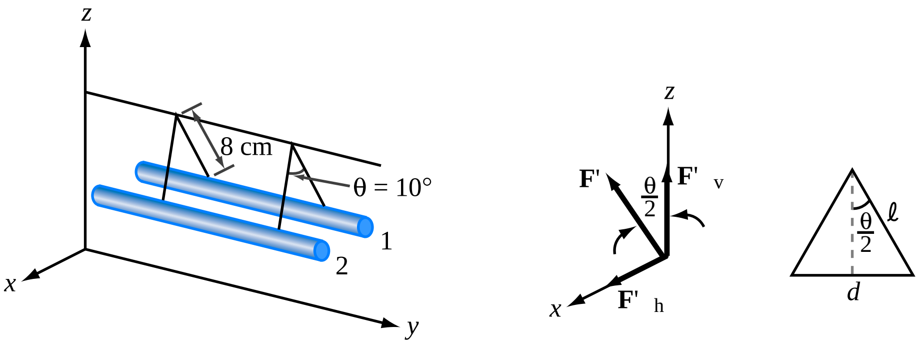
$$\mathbf{F}_{\text{m2}} = I_2 \boldsymbol{\ell} \times \mathbf{B} = I_2(-\hat{\mathbf{z}}b) \times \left(\hat{\mathbf{y}} \frac{\mu_0 I_1}{2\pi x} \right) \bigg|_{x=a+d} = \hat{\mathbf{x}} \frac{\mu_0 I_1 I_2 b}{2\pi(a+d)}.$$

The other two sides do not contribute any net forces to the loop because they are equal in magnitude and opposite in direction. Therefore, the total force on the loop is

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_{\mathrm{m1}} + \mathbf{F}_{\mathrm{m2}} \\ &= -\hat{\mathbf{x}}\frac{\mu_0 I_1 I_2 b}{2\pi d} + \hat{\mathbf{x}}\frac{\mu_0 I_1 I_2 b}{2\pi(a+d)} \\ &= -\hat{\mathbf{x}}\frac{\mu_0 I_1 I_2 ab}{2\pi d(a+d)} \\ &= -\hat{\mathbf{x}}\frac{4\pi\times10^{-7}\times20\times30\times0.2\times0.5}{2\pi\times0.1\times0.3} = -\hat{\mathbf{x}}0.4 \quad \text{(mN)}.\end{aligned}$$

The force is pulling the loop toward the wire.

Problem 5.16 In the arrangement shown in Fig. 5-41 (P5.16), each of the two long, parallel conductors carries a current I, is supported by 8-cm-long strings, and has a mass per unit length of 1.2 g/cm. Due to the repulsive force acting on the conductors, the angle θ between the supporting strings is 10°. Determine the magnitude of I and the relative directions of the currents in the two conductors.



(a) (b) (c)

Figure P5.16: Parallel conductors supported by strings (Problem 5.16).

Solution: While the vertical component of the tension in the strings is counteracting the force of gravity on the wires, the horizontal component of the tension in the strings is counteracting the magnetic force, which is pushing the wires apart. According to Section 5-3, the magnetic force is repulsive when the currents are in opposite directions.

Figure P5.16(b) shows forces on wire 1 of part (a). The quantity $\mathbf{F}'$ is the tension force per unit length of wire due to the mass per unit length $m' = 1.2$ g/cm $= 0.12$

kg/m. The vertical component of $\mathbf{F}'$ balances out the gravitational force,

$$F'_{\rm v} = m'g, \tag{19}$$

where $g = 9.8$ (m/s^2). But

$$F'_{\rm v} = F' \cos(\theta/2). \tag{20}$$

Hence,

$$F' = \frac{m'g}{\cos(\theta/2)}. \tag{21}$$

The horizontal component of $\mathbf{F}'$ must be equal to the repulsion magnitude force given by Eq. (5.42):

$$F'_{\rm h} = \frac{\mu_0 I^2}{2\pi d} = \frac{\mu_0 I^2}{2\pi[2\ell \sin(\theta/2)]}, \tag{22}$$

where d is the spacing between the wires and ℓ is the length of the string, as shown in Fig. P5.16(c). From Fig. 5.16(b),

$$F'_{\rm h} = F' \sin(\theta/2) = \frac{m'g}{\cos(\theta/2)} \sin(\theta/2) = m'g \tan(\theta/2). \tag{23}$$

Equating Eqs. (22) and (23) and then solving for I, we have

$$I = \sin(\theta/2) \sqrt{\frac{4\pi\ell m'g}{\mu_0 \cos(\theta/2)}} = \sin 5^\circ \sqrt{\frac{4\pi \times 0.08 \times 0.12 \times 9.8}{4\pi \times 10^{-7} \cos 5^\circ}} = 84.8 \quad \text{(A)}.$$

Problem 5.17 An infinitely long, thin conducting sheet of width w along the x-direction lies in the x–y plane and carries a current I in the $-y$-direction. Determine (a) the magnetic field at a point P midway between the edges of the sheet and at a height h above it (Fig. 5-42 (P5.17)), and then (b) determine the force per unit length exerted on an infinitely long wire passing through point P and parallel to the sheet if the current through the wire is equal in magnitude but opposite in direction to that carried by the sheet.

Solution:

(a) The sheet can be considered to consist of a large number of infinitely long but narrow wires each dx wide lying next to each other, with each carrying a current

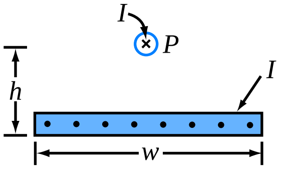


Figure P5.17: A linear current source above a current sheet (Problem 5.17).

$I_x = I\,dx/w$. If we choose the coordinate system shown in Fig. P5.17, the wire at a distance x from the origin is at a distance vector $\mathbf{R}$ from point P, with

$$\mathbf{R} = -\hat{\mathbf{x}}x + \hat{\mathbf{z}}h.$$

Equation (5.30) provides an expression for the magnetic field due to an infinitely long wire carrying a current I as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \hat{\boldsymbol{\phi}}\,\frac{I}{2\pi r}\,.$$

We now need to adapt this expression to the present situation by replacing I with $I_x = I\,dx/w$, replacing r with $R = (x^2+h^2)^{1/2}$, and by assigning the proper direction for the magnetic field. From the Biot–Savart law, the direction of $\mathbf{H}$ is governed by $\mathbf{l}\times\mathbf{R}$, where $\mathbf{l}$ is the direction of current flow. In the present case, $\mathbf{l}$ is out of the page, which is the $-\hat{\mathbf{y}}$ direction. Hence, the direction of the field is

$$\frac{\mathbf{l}\times\mathbf{R}}{|\mathbf{l}\times\mathbf{R}|} = \frac{-\hat{\mathbf{y}}\times(-\hat{\mathbf{x}}x+\hat{\mathbf{z}}h)}{|-\hat{\mathbf{y}}\times(-\hat{\mathbf{x}}x+\hat{\mathbf{z}}h)|} = \frac{-(\hat{\mathbf{x}}h+\hat{\mathbf{z}}x)}{(x^2+h^2)^{1/2}}\,.$$

Therefore, the field $d\mathbf{H}$ due to current I_x is

$$d\mathbf{H} = \frac{-(\hat{\mathbf{x}}h+\hat{\mathbf{z}}x)}{(x^2+h^2)^{1/2}}\,\frac{I_x}{2\pi R} = \frac{-(\hat{\mathbf{x}}h+\hat{\mathbf{z}}x)I\,dx}{2\pi w(x^2+h^2)}\,,$$

and the total field is

$$\begin{aligned}
\mathbf{H}(0,0,h) &= \int_{x=-w/2}^{w/2} -(\hat{\mathbf{x}}h+\hat{\mathbf{z}}x)\frac{I\,dx}{2\pi w(x^2+h^2)} \\
&= \frac{-I}{2\pi w}\int_{x=-w/2}^{w/2}(\hat{\mathbf{x}}h+\hat{\mathbf{z}}x)\frac{dx}{x^2+h^2} \\
&= \frac{-I}{2\pi w}\left(\hat{\mathbf{x}}h\int_{x=-w/2}^{w/2}\frac{dx}{x^2+h^2} + \hat{\mathbf{z}}\int_{x=-w/2}^{w/2}\frac{x\,dx}{x^2+h^2}\right)
\end{aligned}$$

$$= \frac{-I}{2\pi w}\left(\hat{\mathbf{x}}h\left(\frac{1}{h}\tan^{-1}\left(\frac{x}{h}\right)\right)\Bigg|_{x=-w/2}^{w/2} + \hat{\mathbf{z}}\left(\tfrac{1}{2}\ln\left(x^2+h^2\right)\right)\Big|_{x=-w/2}^{w/2}\right)$$

$$= -\hat{\mathbf{x}}\frac{I}{\pi w}\tan^{-1}\left(\frac{w}{2h}\right) \quad \text{(A/m).}$$

At P in Fig. P5.17, the field is pointing to the left. The z-component could have been assumed zero with a symmetry argument. An alternative solution is to employ Eq. (5.24a) directly.

(b) From Eq. (5.9), a differential force is of the form $d\mathbf{F}_{\text{m}} = I\,d\mathbf{l}\times\mathbf{B}$ or, assuming $d\mathbf{l} = \hat{\mathbf{a}}_\ell\,d\ell$, the force per unit length is given by

$$\mathbf{F}'_{\text{m}} = \frac{\partial\mathbf{F}_{\text{m}}}{\partial\ell} = I\hat{\mathbf{a}}_\ell\times\mathbf{B} = I\hat{\mathbf{y}}\times\left(-\hat{\mathbf{x}}\frac{\mu_0 I}{\pi w}\tan^{-1}\left(\frac{w}{2h}\right)\right) = \hat{\mathbf{z}}\frac{\mu_0 I^2}{\pi w}\tan^{-1}\left(\frac{w}{2h}\right) \quad \text{(N).}$$

The force is repulsive; the wire is experiencing a force pushing it up.

Problem 5.18 Three long, parallel wires are arranged as shown in Fig. 5-43 (P5.18(a)). Determine the force per unit length acting on the wire carrying I_3.

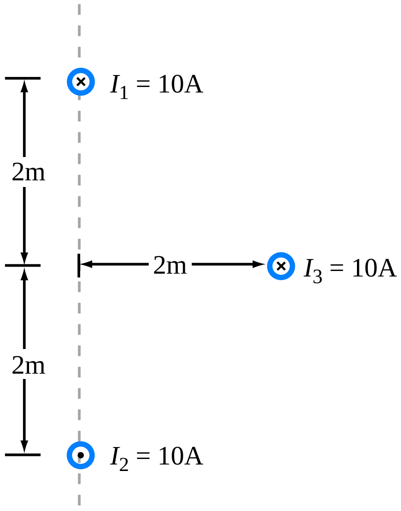


Figure P5.18: (a) Three parallel wires of Problem 5.18.

Solution: Since I_1 and I_2 are equal in magnitude and opposite in direction, and equidistant from I_3, our intuitive answer might be that the net force on I_3 is zero. As

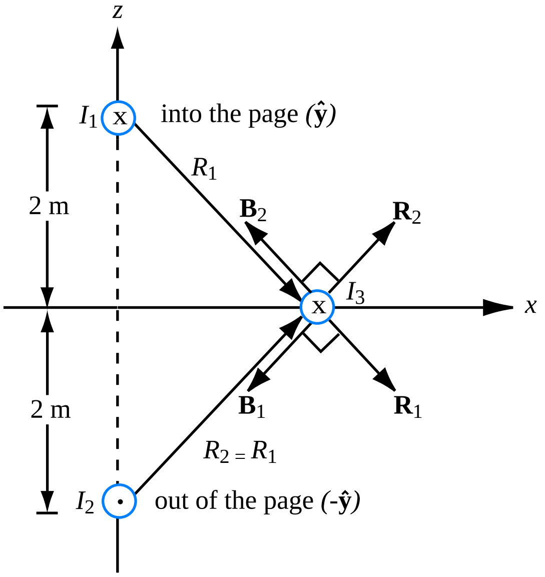


Figure P5.18: (b) **B** fields due to I_1 and I_2 at location of I_3.

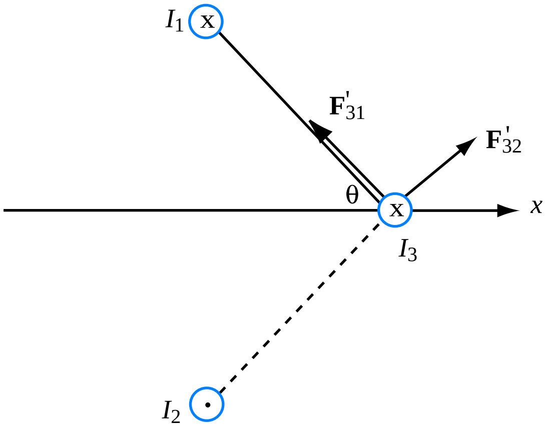


Figure P5.18: (c) Forces acting on I_3.

we will see, that's not the correct answer. The field due to I_1 (which is along $\hat{\mathbf{y}}$) at location of I_3 is

$$\mathbf{B}_1 = \hat{\mathbf{b}}_1 \frac{\mu_0 I_1}{2\pi R_1}$$

where $\hat{\mathbf{b}}_1$ is the unit vector in the direction of $\mathbf{B}_1$ shown in the figure, which is perpendicular to $\hat{\mathbf{R}}_1$. The force per unit length exerted on I_3 is

$$\mathbf{F}'_{31} = \frac{\mu_0 I_1 I_3}{2\pi R_1}(\hat{\mathbf{y}} \times \hat{\mathbf{b}}_1) = -\hat{\mathbf{R}}_1 \frac{\mu_0 I_1 I_3}{2\pi R_1}.$$

Similarly, the force per unit length excited on I_3 by the field due to I_2 (which is along $-\hat{\mathbf{y}}$) is

$$\mathbf{F}'_{32} = \hat{\mathbf{R}}_2 \frac{\mu_0 I_2 I_3}{2\pi R_2}.$$

The two forces have opposite components along $\hat{\mathbf{x}}$ and equal components along $\hat{\mathbf{z}}$. Hence, with $R_1 = R_2 = \sqrt{8}$ m and $\theta = \sin^{-1}(2/\sqrt{8}) = \sin^{-1}(1/\sqrt{2}) = 45°$,

$$\begin{aligned}\mathbf{F}'_3 = \mathbf{F}'_{31} + \mathbf{F}'_{32} &= \hat{\mathbf{z}}\left(\frac{\mu_0 I_1 I_3}{2\pi R_1} + \frac{\mu_0 I_2 I_3}{2\pi R_2}\right)\sin\theta \\ &= \hat{\mathbf{z}}2\left(\frac{4\pi \times 10^{-7} \times 10 \times 20}{2\pi \times \sqrt{8}}\right) \times \frac{1}{\sqrt{2}} = \hat{\mathbf{z}}2 \times 10^{-5} \text{ N/m.}\end{aligned}$$

Problem 5.19 A square loop placed as shown in Fig. 5-44 (P5.19) has 2-m sides and carries a current $I_1 = 5$ A. If a straight, long conductor carrying a current $I_2 = 10$ A is introduced and placed just above the midpoints of two of the loop's sides, determine the net force acting on the loop.

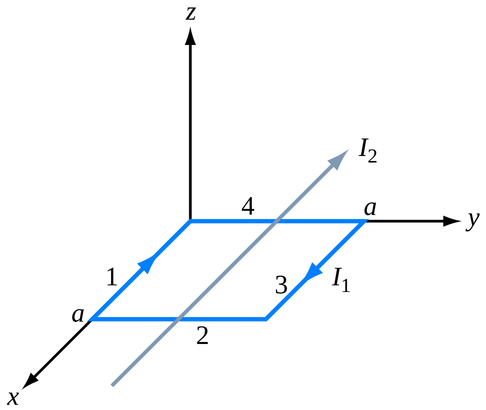


Figure P5.19: Long wire carrying current I_2, just above a square loop carrying I_1 (Problem 5.19).

Solution: Since I_2 is just barely above the loop, we can treat it as if it's in the same plane as the loop. For side 1, I_1 and I_2 are in the same direction, hence the force on side 1 is attractive. That is,

$$\mathbf{F}_1 = \hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2 a}{2\pi(a/2)} = \hat{\mathbf{y}}\,\frac{4\pi\times10^{-7}\times5\times10\times2}{2\pi\times1} = \hat{\mathbf{y}}\,2\times10^{-5}\text{ N}.$$

I_1 and I_2 are in opposite directions for side 3. Hence, the force on side 3 is repulsive, which means it is also along $\hat{\mathbf{y}}$. That is, $\mathbf{F}_3 = \mathbf{F}_1$.

The net forces on sides 2 and 4 are zero. Total net force on the loop is

$$\mathbf{F} = 2\mathbf{F}_1 = \hat{\mathbf{y}}\,4\times10^{-5}\text{ N}.$$

Section 5-4: Gauss's Law for Magnetism and Ampère's Law

Problem 5.20 Current I flows along the positive z-direction in the inner conductor of a long coaxial cable and returns through the outer conductor. The inner conductor has radius a, and the inner and outer radii of the outer conductor are b and c, respectively.

(a) Determine the magnetic field in each of the following regions: $0 \le r \le a$, $a \le r \le b$, $b \le r \le c$, and $r \ge c$.

(b) Plot the magnitude of $\mathbf{H}$ as a function of r over the range from $r = 0$ to $r = 10$ cm, given that $I = 10$ A, $a = 2$ cm, $b = 4$ cm, and $c = 5$ cm.

Solution:

(a) Following the solution to Example 5-5, the magnetic field in the region $r < a$,

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{rI}{2\pi a^2},$$

and in the region $a < r < b$,

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{I}{2\pi r}.$$

The total area of the outer conductor is $A = \pi(c^2 - b^2)$ and the fraction of the area of the outer conductor enclosed by a circular contour centered at $r = 0$ in the region $b < r < c$ is

$$\frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)} = \frac{r^2 - b^2}{c^2 - b^2}.$$

The total current enclosed by a contour of radius r is therefore

$$I_{\text{enclosed}} = I\left(1 - \frac{r^2 - b^2}{c^2 - b^2}\right) = I\,\frac{c^2 - r^2}{c^2 - b^2},$$

and the resulting magnetic field is

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{I_{\text{enclosed}}}{2\pi r} = \hat{\boldsymbol{\phi}}\frac{I}{2\pi r}\left(\frac{c^2-r^2}{c^2-b^2}\right).$$

For $r > c$, the total enclosed current is zero: the total current flowing on the inner conductor is equal to the total current flowing on the outer conductor, but they are flowing in opposite directions. Therefore, $\mathbf{H} = 0$.

(b) See Fig. P5.20.

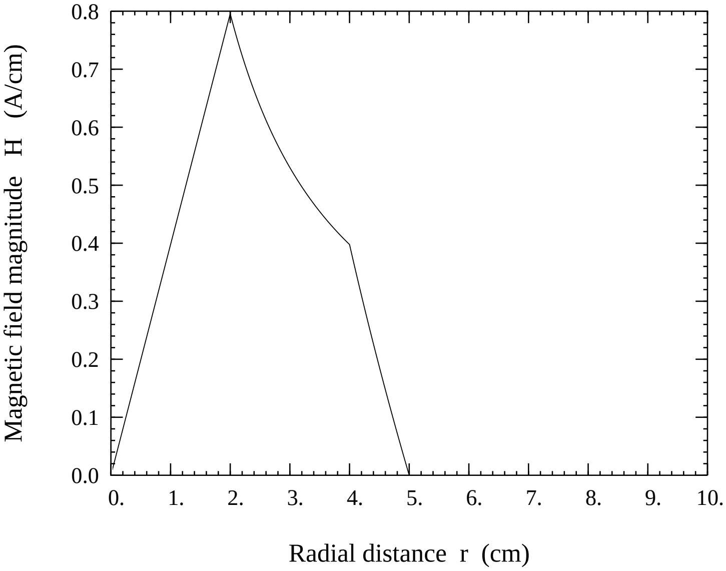


Figure P5.20: Problem 5.20(b).

Problem 5.21 A long cylindrical conductor whose axis is coincident with the z-axis has a radius a and carries a current characterized by a current density $\mathbf{J} = \hat{\mathbf{z}}J_0/r$, where J_0 is a constant and r is the radial distance from the cylinder's axis. Obtain an expression for the magnetic field $\mathbf{H}$ for (a) $0 \le r \le a$ and (b) $r > a$.

Solution: This problem is very similar to Example 5-5.

(a) For $0 \le r_1 \le a$, the total current flowing within the contour C_1 is

$$I_1 = \iint \mathbf{J}\cdot d\mathbf{s} = \int_{\phi=0}^{2\pi}\int_{r=0}^{r_1}\left(\frac{\hat{\mathbf{z}}J_0}{r}\right)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi) = 2\pi\int_{r=0}^{r_1} J_0\,dr = 2\pi r_1 J_0.$$

Therefore, since $I_1 = 2\pi r_1 H_1,\ H_1 = J_0$ within the wire and $\mathbf{H}_1 = \hat{\boldsymbol{\phi}} J_0$.

(b) For $r \geq a$, the total current flowing within the contour is the total current flowing within the wire:

$$I = \iint \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^{a} \left(\frac{\hat{\mathbf{z}} J_0}{r} \right) \cdot (\hat{\mathbf{z}} r\, dr\, d\phi) = 2\pi \int_{r=0}^{a} J_0\, dr = 2\pi a J_0.$$

Therefore, since $I = 2\pi r H_2,\ H_2 = J_0 a/r$ within the wire and $\mathbf{H}_2 = \hat{\boldsymbol{\phi}} J_0 (a/r)$.

Problem 5.22 Repeat Problem 5.21 for a current density $\mathbf{J} = \hat{\mathbf{z}} J_0 e^{-r}$.

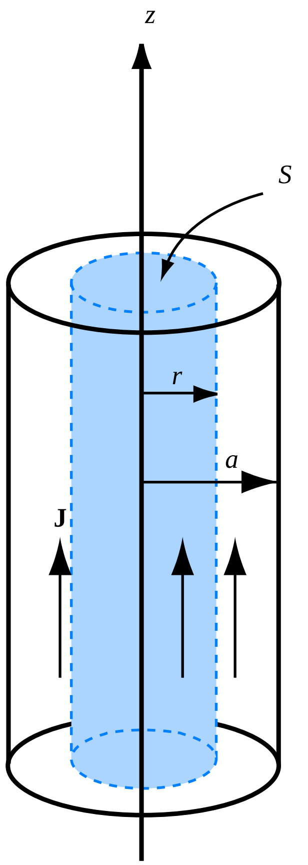


Figure P5.22: Cylindrical current.

Solution:

(a) For $r \le a$, Ampère's law is

$$\begin{aligned}
\oint_c \mathbf{H}\cdot d\mathbf{l} &= I = \int_S \mathbf{J}\cdot d\mathbf{s},\\
\hat{\boldsymbol{\phi}} H\cdot\hat{\boldsymbol{\phi}}2\pi r &= \int_0^r \mathbf{J}\cdot d\mathbf{s} = \int_0^r \hat{\mathbf{z}}J_0 e^{-r}\cdot\hat{\mathbf{z}}2\pi r\,dr,\\
2\pi r H &= 2\pi J_0\int_0^r re^{-r}\,dr\\
&= 2\pi J_0[-e^{-r}(r+1)]_0^r = 2\pi J_0[1-e^{-r}(r+1)].
\end{aligned}$$

Hence,

$$\mathbf{H} = \hat{\boldsymbol{\phi}}H = \hat{\boldsymbol{\phi}}\frac{J_0}{r}[1-e^{-r}(r+1)], \qquad \text{for } r \le a.$$

(b) For $r \ge a$,

$$\begin{aligned}
2\pi r H &= 2\pi J_0[-e^{-r}(r+1)]_0^a = 2\pi J_0[1-e^{-a}(a+1)],\\
\mathbf{H} &= \hat{\boldsymbol{\phi}}H = \hat{\boldsymbol{\phi}}\frac{J_0}{r}\left[1-e^{-a}(a+1)\right], \qquad r \ge a.
\end{aligned}$$

Problem 5.23 In a certain conducting region, the magnetic field is given in cylindrical coordinates by

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\frac{4}{r}[1-(1+3r)e^{-3r}].$$

Find the current density $\mathbf{J}$.

Solution:

$$\begin{aligned}
\mathbf{J} = \nabla\times\mathbf{H} &= \hat{\mathbf{z}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{4}{r}[1-(1+3r)e^{-3r}]\right)\\
&= \hat{\mathbf{z}}\frac{1}{r}[12e^{-2r}(1+2r)-12e^{-2r}] = \hat{\mathbf{z}}24e^{-3r}\ \text{A/m}^2.
\end{aligned}$$

Section 5-5: Magnetic Potential

Problem 5.24 With reference to Fig. 5-10, (a) derive an expression for the vector magnetic potential $\mathbf{A}$ at a point P located at a distance r from the wire in the x–y plane, and then (b) derive $\mathbf{B}$ from $\mathbf{A}$. Show that your result is identical with the expression given by Eq. (5.29), which was derived by applying the Biot–Savart law.

Solution:

(a) From the text immediately following Eq. (5.65), that equation may take the form

$$\begin{aligned}\mathbf{A} &= \frac{\mu}{4\pi}\int_{\ell'}\frac{I}{R'}d\mathbf{l}' = \frac{\mu_0}{4\pi}\int_{z'=-\ell/2}^{\ell/2}\frac{I}{\sqrt{z'^2+r^2}}\,\hat{\mathbf{z}}\,dz' \\ &= \frac{\mu_0}{4\pi}\left(\hat{\mathbf{z}}I\ln\left(z'+\sqrt{z'^2+r^2}\right)\right)\bigg|_{z'=-\ell/2}^{\ell/2} \\ &= \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\ln\left(\frac{\ell/2+\sqrt{(\ell/2)^2+r^2}}{-\ell/2+\sqrt{(-\ell/2)^2+r^2}}\right) \\ &= \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\ln\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right).\end{aligned}$$

(b) From Eq. (5.53),

$$\begin{aligned}\mathbf{B} &= \nabla\times\mathbf{A} \\ &= \nabla\times\left(\hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\ln\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right)\right) \\ &= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\frac{\partial}{\partial r}\ln\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right) \\ &= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{-\ell+\sqrt{\ell^2+4r^2}}{\ell+\sqrt{\ell^2+4r^2}}\right)\frac{\partial}{\partial r}\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right) \\ &= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{-\ell+\sqrt{\ell^2+4r^2}}{\ell+\sqrt{\ell^2+4r^2}}\right) \\ &\times\left(\frac{(-\ell+\sqrt{\ell^2+4r^2})\frac{\partial}{\partial r}(\ell+\sqrt{\ell^2+4r^2})-(\ell+\sqrt{\ell^2+4r^2})\frac{\partial}{\partial r}(-\ell+\sqrt{\ell^2+4r^2})}{(-\ell+\sqrt{\ell^2+4r^2})^2}\right) \\ &= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{(-\ell+\sqrt{\ell^2+4r^2})-(\ell+\sqrt{\ell^2+4r^2})}{(-\ell+\sqrt{\ell^2+4r^2})(\ell+\sqrt{\ell^2+4r^2})}\right)\frac{4r}{\sqrt{\ell^2+4r^2}} \\ &= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{-2\ell}{4r^2}\right)\frac{4r}{\sqrt{\ell^2+4r^2}} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I\ell}{2\pi r\sqrt{\ell^2+4r^2}} \quad \text{(T)}.\end{aligned}$$

which is the same as Eq. (5.29).

Problem 5.25 In a given region of space, the vector magnetic potential is given by $\mathbf{A} = \hat{\mathbf{x}}5\cos\pi y + \hat{\mathbf{z}}(2+\sin\pi x)$ (Wb/m).

- **(a)** Determine **B**.
- **(b)** Use Eq. (5.66) to calculate the magnetic flux passing through a square loop with 0.25-m-long edges if the loop is in the *x*–*y* plane, its center is at the origin, and its edges are parallel to the *x*- and *y*-axes.
- **(c)** Calculate Φ again using Eq. (5.67).

Solution:

(a) From Eq. (5.53), $\mathbf{B} = \nabla \times \mathbf{A} = \hat{\mathbf{z}}5\pi \sin \pi y - \hat{\mathbf{y}}\pi \cos \pi x$.

(b) From Eq. (5.66),

$$\begin{aligned}
\Phi = \iint \mathbf{B} \cdot d\mathbf{s} &= \int_{y=-0.125\text{ m}}^{0.125\text{ m}} \int_{x=-0.125\text{ m}}^{0.125\text{ m}} (\hat{\mathbf{z}}5\pi \sin \pi y - \hat{\mathbf{y}}\pi \cos \pi x) \cdot (\hat{\mathbf{z}}\, dx\, dy) \\
&= \left(\left(-5\pi x \frac{\cos \pi y}{\pi} \right) \Big|_{x=-0.125}^{0.125} \right) \Bigg|_{y=-0.125}^{0.125} \\
&= \frac{-5}{4} \left(\cos\left(\frac{\pi}{8}\right) - \cos\left(\frac{-\pi}{8}\right) \right) = 0.
\end{aligned}$$

(c) From Eq. (5.67), $\Phi = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}$, where C is the square loop in the *x*-*y* plane with sides of length 0.25 m centered at the origin. Thus, the integral can be written as

$$\Phi = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell} = S_{\text{front}} + S_{\text{back}} + S_{\text{left}} + S_{\text{right}},$$

where S_{front}, S_{back}, S_{left}, and S_{right} are the sides of the loop.

$$\begin{aligned}
S_{\text{front}} &= \int_{x=-0.125}^{0.125} (\hat{\mathbf{x}}5 \cos \pi y + \hat{\mathbf{z}}(2 + \sin \pi x))\big|_{y=-0.125} \cdot (\hat{\mathbf{x}}\, dx) \\
&= \int_{x=-0.125}^{0.125} 5 \cos \pi y\big|_{y=-0.125}\, dx \\
&= \left((5x \cos \pi y)\big|_{y=-0.125} \right) \Big|_{x=-0.125}^{0.125} = \frac{5}{4} \cos\left(\frac{-\pi}{8}\right) = \frac{5}{4} \cos\left(\frac{\pi}{8}\right), \\
S_{\text{back}} &= \int_{x=-0.125}^{0.125} (\hat{\mathbf{x}}5 \cos \pi y + \hat{\mathbf{z}}(2 + \sin \pi x))\big|_{y=0.125} \cdot (-\hat{\mathbf{x}}\, dx) \\
&= -\int_{x=-0.125}^{0.125} 5 \cos \pi y\big|_{y=0.125}\, dx \\
&= \left((-5x \cos \pi y)\big|_{y=0.125} \right) \Big|_{x=-0.125}^{0.125} = -\frac{5}{4} \cos\left(\frac{\pi}{8}\right), \\
S_{\text{left}} &= \int_{y=-0.125}^{0.125} (\hat{\mathbf{x}}5 \cos \pi y + \hat{\mathbf{z}}(2 + \sin \pi x))\big|_{x=-0.125} \cdot (-\hat{\mathbf{y}}\, dy)
\end{aligned}$$

$$= -\int_{y=-0.125}^{0.125} 0|_{x=-0.125}\, dy = 0,$$

$$S_{\text{right}} = \int_{y=-0.125}^{0.125} (\hat{\mathbf{x}}5\cos\pi y + \hat{\mathbf{z}}(2+\sin\pi x))|_{x=0.125} \cdot (\hat{\mathbf{y}}\, dy)$$

$$= \int_{y=-0.125}^{0.125} 0|_{x=0.125}\, dy = 0.$$

Thus,

$$\Phi = \oint_c \mathbf{A}\cdot d\boldsymbol{\ell} = S_{\text{front}} + S_{\text{back}} + S_{\text{left}} + S_{\text{right}} = \frac{5}{4}\cos\left(\frac{\pi}{8}\right) - \frac{5}{4}\cos\left(\frac{\pi}{8}\right) + 0 + 0 = 0.$$

Problem 5.26 A uniform current density given by

$$\mathbf{J} = \hat{\mathbf{z}}J_0 \quad (\text{A/m}^2),$$

gives rise to a vector magnetic potential

$$\mathbf{A} = -\hat{\mathbf{z}}\frac{\mu_0 J_0}{4}(x^2+y^2) \quad (\text{Wb/m}).$$

- **(a)** Apply the vector Poisson's equation to confirm the above statement.
- **(b)** Use the expression for **A** to find **H**.
- **(c)** Use the expression for **J** in conjunction with Ampère's law to find **H**. Compare your result with that obtained in part (b).

Solution:
(a)

$$\nabla^2\mathbf{A} = \hat{\mathbf{x}}\nabla^2 A_x + \hat{\mathbf{y}}\nabla^2 A_y + \hat{\mathbf{z}}\nabla^2 A_z = \hat{\mathbf{z}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left[-\mu_0\frac{J_0}{4}(x^2+y^2)\right]$$
$$= -\hat{\mathbf{z}}\mu_0\frac{J_0}{4}(2+2) = -\hat{\mathbf{z}}\mu_0 J_0.$$

Hence, $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$ is verified.
(b)

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} = \frac{1}{\mu_0}\left[\hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$
$$= \frac{1}{\mu_0}\left(\hat{\mathbf{x}}\frac{\partial A_z}{\partial y} - \hat{\mathbf{y}}\frac{\partial A_z}{\partial x}\right)$$
$$= \frac{1}{\mu_0}\left[\hat{\mathbf{x}}\frac{\partial}{\partial y}\left(-\mu_0\frac{J_0}{4}(x^2+y^2)\right) - \hat{\mathbf{y}}\frac{\partial}{\partial x}\left(-\mu_0\frac{J_0}{4}(x^2+y^2)\right)\right]$$
$$= -\hat{\mathbf{x}}\frac{J_0 y}{2} + \hat{\mathbf{y}}\frac{J_0 x}{2} \quad (\text{A/m}).$$

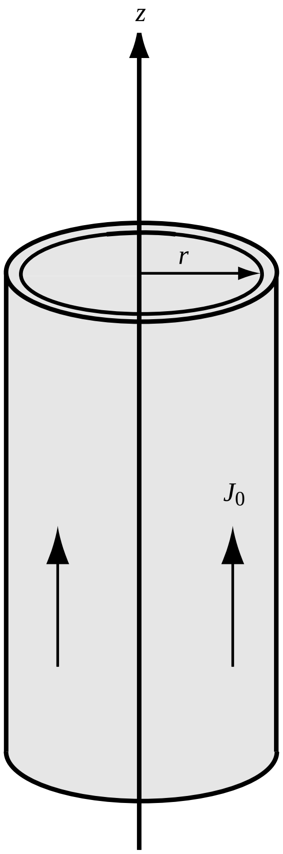


Figure P5.26: Current cylinder of Problem 5.26.

(c)

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = I = \int_S \mathbf{J}\cdot d\mathbf{s},$$
$$\hat{\boldsymbol{\phi}} H_\phi \cdot \hat{\boldsymbol{\phi}} 2\pi r = J_0 \cdot \pi r^2,$$
$$\mathbf{H} = \hat{\boldsymbol{\phi}} H_\phi = \hat{\boldsymbol{\phi}} J_0 \frac{r}{2}.$$

We need to convert the expression from cylindrical to Cartesian coordinates. From Table 3-2,

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi = -\hat{\mathbf{x}}\frac{y}{\sqrt{x^2+y^2}} + \hat{\mathbf{y}}\frac{x}{\sqrt{x^2+y^2}},$$
$$r = \sqrt{x^2+y^2}.$$

Hence

$$\mathbf{H} = \left(-\hat{\mathbf{x}}\frac{y}{\sqrt{x^2+y^2}} + \hat{\mathbf{y}}\frac{x}{\sqrt{x^2+y^2}}\right)\cdot\frac{J_0}{2}\sqrt{x^2+y^2} = -\hat{\mathbf{x}}\frac{yJ_0}{2} + \hat{\mathbf{y}}\frac{xJ_0}{2},$$

which is identical with the result of part (b).

Problem 5.27 A thin current element extending between $z = -L/2$ and $z = L/2$ carries a current I along $+\hat{\mathbf{z}}$ through a circular cross section of radius a.

(a) Find **A** at a point P located very far from the origin (assume R is so much larger than L that point P may be considered to be at approximately the same distance from every point along the current element).

(b) Determine the corresponding **H**.

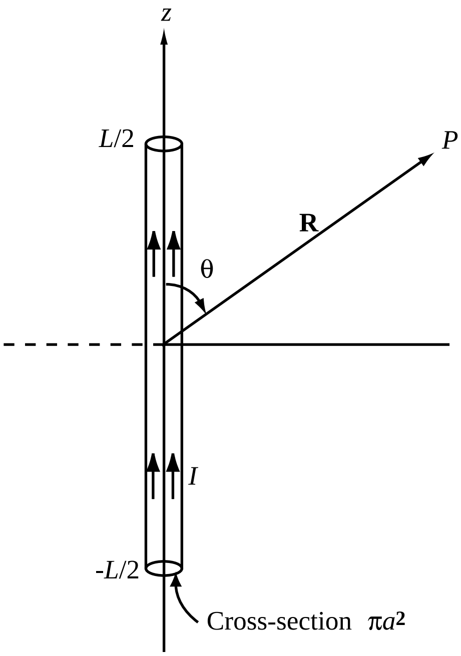


Figure P5.27: Current element of length L observed at distance $R \gg L$.

Solution:

(a) Since $R \gg L$, we can assume that P is approximately equidistant from all segments of the current element. Hence, with R treated as constant, (5.65) gives

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{\mathcal{V}'}\frac{\mathbf{J}}{R'}\,d\mathcal{V}' = \frac{\mu_0}{4\pi R}\int_{\mathcal{V}'}\hat{\mathbf{z}}\,\frac{I}{(\pi a^2)}\,\pi a^2\,dz = \hat{\mathbf{z}}\,\frac{\mu_0 I}{4\pi R}\int_{-L/2}^{L/2}dz = \hat{\mathbf{z}}\,\frac{\mu_0 IL}{4\pi R}.$$

(b)

$$
\begin{aligned}
\mathbf{H} &= \frac{1}{\mu_0}\nabla\times\mathbf{A} \\
&= \frac{1}{\mu_0}\left[\hat{\mathbf{x}}\frac{\partial A_z}{\partial y}-\hat{\mathbf{y}}\frac{\partial A_z}{\partial x}\right] \\
&= \frac{1}{\mu_0}\left\{\hat{\mathbf{x}}\frac{\partial}{\partial y}\left[\frac{\mu_0 IL}{4\pi}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\right]-\hat{\mathbf{y}}\frac{\partial}{\partial x}\left[\frac{\mu_0 IL}{4\pi}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\right]\right\} \\
&= \frac{IL}{4\pi}\left[\frac{-\hat{\mathbf{x}}y+\hat{\mathbf{y}}x}{(x^2+y^2+z^2)^{3/2}}\right].
\end{aligned}
$$

Section 5-6: Magnetic Properties of Materials

Problem 5.28 In the model of the hydrogen atom proposed by Bohr in 1913, the electron moves around the nucleus at a speed of 2×10^6 m/s in a circular orbit of radius 5×10^{-11} m. What is the magnitude of the magnetic moment generated by the electron's motion?

Solution: From Eq. (5.69), the magnitude of the orbital magnetic moment of an electron is

$$|m_0| = \left|-\tfrac{1}{2}eur\right| = \tfrac{1}{2}\times1.6\times10^{-19}\times2\times10^6\times5\times10^{-11} = 8\times10^{-24} \quad (\text{A}\cdot\text{m}^2).$$

Problem 5.29 Iron contains 8.5×10^{28} atoms/m^3. At saturation, the alignment of the electrons' spin magnetic moments in iron can contribute 1.5 T to the total magnetic flux density **B**. If the spin magnetic moment of a single electron is 9.27×10^{-24} (A·m^2), how many electrons per atom contribute to the saturated field?

Solution: From the first paragraph of Section 5-6.2, the magnetic flux density of a magnetized material is $\mathbf{B}_\text{m} = \mu_0\mathbf{M}$, where $\mathbf{M}$ is the vector sum of the microscopic magnetic dipoles within the material: $\mathbf{M} = N_\text{e}\mathbf{m}_\text{s}$, where $\mathbf{m}_\text{s}$ is the magnitude of the spin magnetic moment of an electron in the direction of the mean magnetization, and N_e is net number of electrons per unit volume contributing to the bulk magnetization. If the number of electrons per atom contributing to the bulk magnetization is n_e, then $N_\text{e} = n_\text{e}N_\text{atoms}$ where $N_\text{atoms} = 8.5\times10^{28}$ atoms/m^3 is the number density of atoms for iron. Therefore,

$$
\begin{aligned}
n_\text{e} = \frac{N_\text{e}}{N_\text{atoms}} = \frac{M}{m_\text{s}N_\text{atoms}} = \frac{B}{\mu_0 m_\text{s}N_\text{atoms}} &= \frac{1.5}{4\pi\times10^{-7}\times9.27\times10^{-24}\times8.5\times10^{28}} \\
&= 1.5 \quad (\text{electrons/atom}).
\end{aligned}
$$

Section 5-7: Magnetic Boundary Conditions

Problem 5.30 The x–y plane separates two magnetic media with magnetic permeabilities μ_1 and μ_2, as shown in Fig. 5-45 (P5.30). If there is no surface current at the interface and the magnetic field in medium 1 is

$$\mathbf{H}_1 = \hat{\mathbf{x}}H_{1x} + \hat{\mathbf{y}}H_{1y} + \hat{\mathbf{z}}H_{1z},$$

find:

(a) $\mathbf{H}_2$,

(b) θ_1 and θ_2, and

(c) evaluate $\mathbf{H}_2$, θ_1, and θ_2 for $H_{1x} = 2$ (A/m), $H_{1y} = 0$, $H_{1z} = 4$ (A/m), $\mu_1 = \mu_0$, and $\mu_2 = 4\mu_0$.

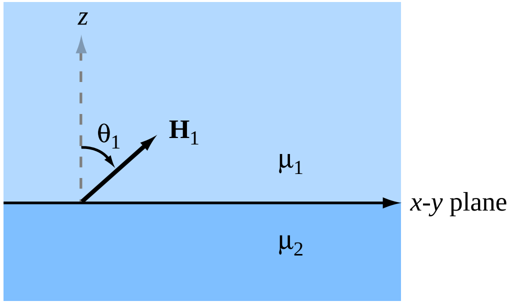


Figure P5.30: Adjacent magnetic media (Problem 5.30).

Solution:

(a) From (5.80),

$$\mu_1 H_{1\text{n}} = \mu_2 H_{2\text{n}},$$

and in the absence of surface currents at the interface, (5.85) states

$$H_{1\text{t}} = H_{2\text{t}}.$$

In this case, $H_{1z} = H_{1\text{n}}$, and H_{1x} and H_{1y} are tangential fields. Hence,

$$\mu_1 H_{1z} = \mu_2 H_{2z},$$
$$H_{1x} = H_{2x},$$
$$H_{1y} = H_{2y},$$

and

$$\mathbf{H}_2 = \hat{\mathbf{x}}H_{1x} + \hat{\mathbf{y}}H_{1y} + \hat{\mathbf{z}}\,\frac{\mu_1}{\mu_2}\,H_{1z}.$$

(b)

$$\begin{aligned}
H_{1\mathrm{t}} &= \sqrt{H_{1x}^2 + H_{1y}^2}\,,\\
\tan\theta_1 &= \frac{H_{1\mathrm{t}}}{H_{1z}} = \frac{\sqrt{H_{1x}^2 + H_{1y}^2}}{H_{1z}}\,,\\
\tan\theta_2 &= \frac{H_{2\mathrm{t}}}{H_{2z}} = \frac{\sqrt{H_{1x}^2 + H_{1y}^2}}{\dfrac{\mu_1}{\mu_2}\,H_{1z}} = \frac{\mu_2}{\mu_1}\tan\theta_1.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{H}_2 &= \hat{\mathbf{x}}\,2 + \hat{\mathbf{z}}\,\frac{1}{4}\cdot 4 = \hat{\mathbf{x}}\,2 + \hat{\mathbf{z}} \quad \text{(A/m)},\\
\theta_1 &= \tan^{-1}\left(\frac{2}{4}\right) = 26.56^\circ,\\
\theta_2 &= \tan^{-1}\left(\frac{2}{1}\right) = 63.44^\circ.
\end{aligned}$$

Problem 5.31 Given that a current sheet with surface current density $\mathbf{J}_\mathrm{s} = \hat{\mathbf{x}}\,8$ (A/m) exists at $y = 0$, the interface between two magnetic media, and $\mathbf{H}_1 = \hat{\mathbf{z}}\,11$ (A/m) in medium 1 $(y > 0)$, determine $\mathbf{H}_2$ in medium 2 $(y < 0)$.

Solution:

$$\begin{aligned}
\mathbf{J}_\mathrm{s} &= \hat{\mathbf{x}}\,8 \text{ A/m},\\
\mathbf{H}_1 &= \hat{\mathbf{z}}\,11 \text{ A/m}.
\end{aligned}$$

$\mathbf{H}_1$ is tangential to the boundary, and therefore $\mathbf{H}_2$ is also. With $\hat{\mathbf{n}}_2 = \hat{\mathbf{y}}$, from Eq. (5.84), we have

$$\begin{aligned}
\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{J}_\mathrm{s},\\
\hat{\mathbf{y}} \times (\hat{\mathbf{z}}\,11 - \mathbf{H}_2) &= \hat{\mathbf{x}}8,\\
\hat{\mathbf{x}}\,11 - \hat{\mathbf{y}} \times \mathbf{H}_2 &= \hat{\mathbf{x}}\,8,
\end{aligned}$$

or

$$\hat{\mathbf{y}} \times \mathbf{H}_2 = \hat{\mathbf{x}}\,3,$$

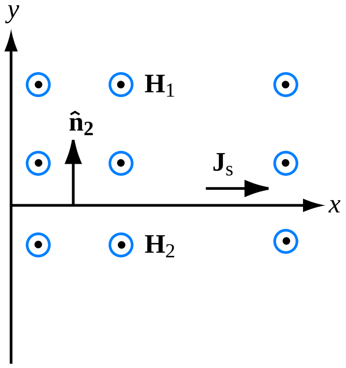


Figure P5.31: Adjacent magnetic media with $\mathbf{J}_s$ on boundary.

which implies that $\mathbf{H}_2$ does not have an x-component. Also, since $\mu_1 H_{1y} = \mu_2 H_{2y}$ and $\mathbf{H}_1$ does not have a y-component, it follows that $\mathbf{H}_2$ does not have a y-component either. Consequently, we conclude that

$$\mathbf{H}_2 = \hat{\mathbf{z}}\,3.$$

Problem 5.32 In Fig. 5-46 (P5.32), the plane defined by $x - y = 1$ separates medium 1 of permeability μ_1 from medium 2 of permeability μ_2. If no surface current exists on the boundary and

$$\mathbf{B}_1 = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 \quad \text{(T)},$$

find $\mathbf{B}_2$ and then evaluate your result for $\mu_1 = 5\mu_2$. Hint: Start out by deriving the equation for the unit vector normal to the given plane.

Solution: We need to find $\hat{\mathbf{n}}_2$. To do so, we start by finding any two vectors in the plane $x - y = 1$, and to do that, we need three non-collinear points in that plane. We choose $(0,-1,0)$, $(1,0,0)$, and $(1,0,1)$.

Vector $\mathbf{A}_1$ is from $(0,-1,0)$ to $(1,0,0)$:

$$\mathbf{A}_1 = \hat{\mathbf{x}}\,1 + \hat{\mathbf{y}}\,1.$$

Vector $\mathbf{A}_2$ is from $(1,0,0)$ to $(1,0,1)$:

$$\mathbf{A}_2 = \hat{\mathbf{z}}\,1.$$

Hence, if we take the cross product $\mathbf{A}_2 \times \mathbf{A}_1$, we end up in a direction normal to the given plane, from medium 2 to medium 1,

$$\hat{\mathbf{n}}_2 = \frac{\mathbf{A}_2 \times \mathbf{A}_1}{|\mathbf{A}_2 \times \mathbf{A}_1|} = \frac{\hat{\mathbf{z}}\,1 \times (\hat{\mathbf{x}}\,1 + \hat{\mathbf{y}}\,1)}{|\mathbf{A}_2 \times \mathbf{A}_1|} = \frac{\hat{\mathbf{y}}\,1 - \hat{\mathbf{x}}\,1}{\sqrt{1+1}} = \frac{\hat{\mathbf{y}}}{\sqrt{2}} - \frac{\hat{\mathbf{x}}}{\sqrt{2}}.$$

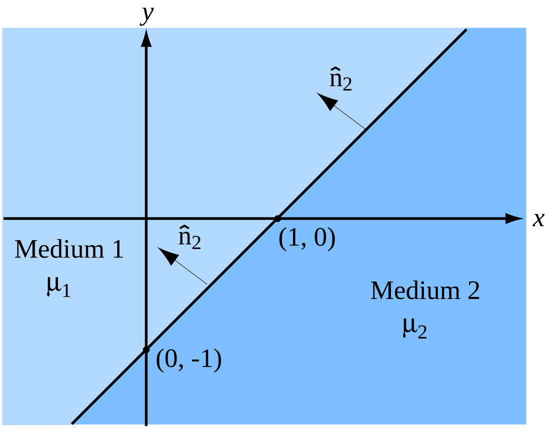


Figure P5.32: Magnetic media separated by the plane $x - y = 1$ (Problem 5.32).

In medium 1, normal component is

$$B_{1n} = \hat{\mathbf{n}}_2 \cdot \mathbf{B}_1 = \left(\frac{\hat{\mathbf{y}}}{\sqrt{2}} - \frac{\hat{\mathbf{x}}}{\sqrt{2}}\right) \cdot (\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3) = \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\mathbf{B}_{1n} = \hat{\mathbf{n}}_2 B_{1n} = \left(\frac{\hat{\mathbf{y}}}{\sqrt{2}} - \frac{\hat{\mathbf{x}}}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} = \frac{\hat{\mathbf{y}}}{2} - \frac{\hat{\mathbf{x}}}{2}.$$

Tangential component is

$$\mathbf{B}_{1t} = \mathbf{B}_1 - \mathbf{B}_{1n} = (\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3) - \left(\frac{\hat{\mathbf{y}}}{2} - \frac{\hat{\mathbf{x}}}{2}\right) = \hat{\mathbf{x}}2.5 + \hat{\mathbf{y}}2.5.$$

Boundary conditions:

$$B_{1n} = B_{2n}, \quad \text{or} \quad \mathbf{B}_{2n} = \frac{\hat{\mathbf{y}}}{2} - \frac{\hat{\mathbf{x}}}{2},$$

$$H_{1t} = H_{2t}, \quad \text{or} \quad \frac{B_{2t}}{\mu_2} = \frac{B_{1t}}{\mu_1}.$$

Hence,

$$\mathbf{B}_{2t} = \frac{\mu_2}{\mu_1}\mathbf{B}_{1t} = \frac{\mu_2}{\mu_1}(\hat{\mathbf{x}}2.5 + \hat{\mathbf{y}}2.5).$$

Finally,

$$\mathbf{B}_2 = \mathbf{B}_{2n} + \mathbf{B}_{2t} = \left(\frac{\hat{\mathbf{y}}}{2} - \frac{\hat{\mathbf{x}}}{2}\right) + \frac{\mu_2}{\mu_1}(\hat{\mathbf{x}}2.5 + \hat{\mathbf{y}}2.5).$$

For $\mu_1 = 5\mu_2$,

$$\mathbf{B}_2 = \hat{\mathbf{y}} \quad \text{(T)}.$$

Problem 5.33 The plane boundary defined by $z = 0$ separates air from a block of iron. If $\mathbf{B}_1 = \hat{\mathbf{x}}4 - \hat{\mathbf{y}}6 + \hat{\mathbf{z}}8$ in air ($z \geq 0$), find $\mathbf{B}_2$ in iron ($z \leq 0$), given that $\mu = 5000\mu_0$ for iron.

Solution: From Eq. (5.2),

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{\mu_1}(\hat{\mathbf{x}}4 - \hat{\mathbf{y}}6 + \hat{\mathbf{z}}8).$$

The z component is the normal component to the boundary at $z = 0$. Therefore, from Eq. (5.79), $B_{2z} = B_{1z} = 8$ while, from Eq. (5.85),

$$H_{2x} = H_{1x} = \frac{1}{\mu_1}4, \qquad H_{2y} = H_{1y} = -\frac{1}{\mu_1}6,$$

or

$$B_{2x} = \mu_2 H_{2x} = \frac{\mu_2}{\mu_1}4, \qquad B_{2y} = \mu_2 H_{2y} = -\frac{\mu_2}{\mu_1}6,$$

where $\mu_2/\mu_1 = \mu_r = 5000$. Therefore,

$$\mathbf{B}_2 = \hat{\mathbf{x}}20000 - \hat{\mathbf{y}}30000 + \hat{\mathbf{z}}8.$$

Problem 5.34 Show that if no surface current densities exist at the parallel interfaces shown in Fig. 5-47 (P5.34), the relationship between θ_4 and θ_1 is independent of μ_2.

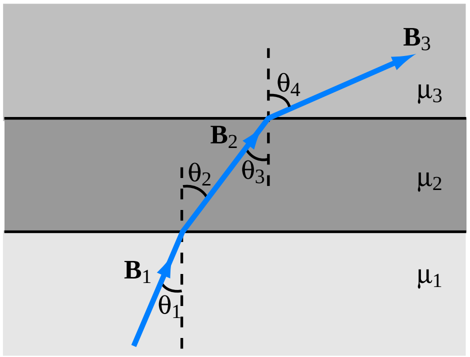


Figure P5.34: Three magnetic media with parallel interfaces (Problem 5.34).

Solution:

$$\tan\theta_1 = \frac{B_{1\mathrm{t}}}{B_{1\mathrm{n}}},$$

and

$$\tan\theta_2 = \frac{B_{2\mathrm{t}}}{B_{2\mathrm{n}}}.$$

But $B_{2\mathrm{n}} = B_{1\mathrm{n}}$ and $\dfrac{B_{2\mathrm{t}}}{\mu_2} = \dfrac{B_{1\mathrm{t}}}{\mu_1}$. Hence,

$$\tan\theta_2 = \frac{B_{1\mathrm{t}}}{B_{1\mathrm{n}}}\frac{\mu_2}{\mu_1} = \frac{\mu_2}{\mu_1}\tan\theta_1.$$

We note that $\theta_2 = \theta_3$ and

$$\tan\theta_4 = \frac{\mu_3}{\mu_2}\tan\theta_3 = \frac{\mu_3}{\mu_2}\tan\theta_2 = \frac{\mu_3}{\mu_2}\frac{\mu_2}{\mu_1}\tan\theta_1 = \frac{\mu_3}{\mu_1}\tan\theta_1,$$

which is independent of μ_2.

Sections 5-8 and 5-9: Inductance and Magnetic Energy

Problem 5.35 Obtain an expression for the self-inductance per unit length for the parallel wire transmission line of Fig. 5-27(a) in terms of a, d, and μ, where a is the radius of the wires, d is the axis-to-axis distance between the wires, and μ is the permeability of the medium in which they reside.

Solution: Let us place the two wires in the x–z plane and orient the current in one of them to be along the $+z$-direction and the current in the other one to be along the $-z$-direction, as shown in Fig. P5.35. From Eq. (5.30), the magnetic field at point $P(x,0,z)$ due to wire 1 is

$$\mathbf{B}_1 = \hat{\boldsymbol{\phi}}\frac{\mu I}{2\pi r} = \hat{\mathbf{y}}\frac{\mu I}{2\pi x},$$

where the permeability has been generalized from free space to any substance with permeability μ, and it has been recognized that in the x-z plane, $\hat{\boldsymbol{\phi}} = \hat{\mathbf{y}}$ and $r = x$ as long as $x > 0$.

Given that the current in wire 2 is opposite that in wire 1, the magnetic field created by wire 2 at point $P(x,0,z)$ is in the same direction as that created by wire 1, and it is given by

$$\mathbf{B}_2 = \hat{\mathbf{y}}\frac{\mu I}{2\pi(d-x)}.$$

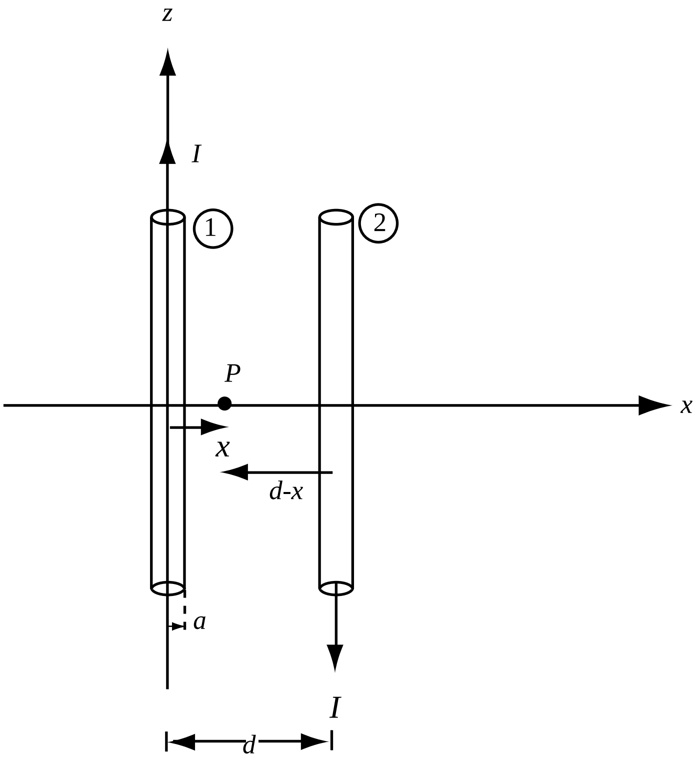


Figure P5.35: Parallel wire transmission line.

Therefore, the total magnetic field in the region between the wires is

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = \hat{\mathbf{y}}\frac{\mu I}{2\pi}\left(\frac{1}{x} + \frac{1}{d-x}\right) = \hat{\mathbf{y}}\frac{\mu I d}{2\pi x(d-x)}\,.$$

From Eq. (5.91), the flux crossing the surface area between the wires over a length l of the wire structure is

$$\begin{aligned}
\Phi = \iint_S \mathbf{B}\cdot d\mathbf{s} &= \int_{z=z_0}^{z_0+l}\int_{x=a}^{d-a}\left(\hat{\mathbf{y}}\frac{\mu I d}{2\pi x(d-x)}\right)\cdot(\hat{\mathbf{y}}\,dx\,dz) \\
&= \frac{\mu I l d}{2\pi}\left(\frac{1}{d}\ln\left(\frac{x}{d-x}\right)\right)\Bigg|_{x=a}^{d-a} \\
&= \frac{\mu I l}{2\pi}\left(\ln\left(\frac{d-a}{a}\right) - \ln\left(\frac{a}{d-a}\right)\right) \\
&= \frac{\mu I l}{2\pi}\times 2\ln\left(\frac{d-a}{a}\right) = \frac{\mu I l}{\pi}\ln\left(\frac{d-a}{a}\right).
\end{aligned}$$

Since the number of 'turns' in this structure is 1, Eq. (5.93) states that the flux linkage is the same as magnetic flux: $\Lambda = \Phi$. Then Eq. (5.94) gives a total inductance over the length l as

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{\mu l}{\pi} \ln\left(\frac{d-a}{a}\right) \quad \text{(H)}.$$

Therefore, the inductance per unit length is

$$L' = \frac{L}{l} = \frac{\mu}{\pi} \ln\left(\frac{d-a}{a}\right) \approx \frac{\mu}{\pi} \ln\left(\frac{d}{a}\right) \quad \text{(H/m)},$$

where the last approximation recognizes that the wires are thin compared to the separation distance (i.e., that $d \gg a$). This has been an implied condition from the beginning of this analysis, where the flux passing through the wires themselves have been ignored. This is the thin-wire limit in Table 2-1 for the two wire line.

Problem 5.36 A solenoid with a length of 20 cm and a radius of 5 cm consists of 400 turns and carries a current of 12 A. If $z = 0$ represents the midpoint of the solenoid, generate a plot for $|\mathbf{H}(z)|$ as a function of z along the axis of the solenoid for the range -20 cm $\leq z \leq 20$ cm in 1-cm steps.

Solution: Let the length of the solenoid be $l = 20$ cm. From Eq. (5.88a) and Eq. (5.88b), $z = a \tan\theta$ and $a^2 + t^2 = a^2 \sec^2\theta$, which implies that $z/\sqrt{z^2 + a^2} = \sin\theta$. Generalizing this to an arbitrary observation point z' on the axis of the solenoid, $(z - z')/\sqrt{(z - z')^2 + a^2} = \sin\theta$. Using this in Eq. (5.89),

$$\begin{aligned}
\mathbf{H}(0,0,z') &= \frac{\mathbf{B}}{\mu} = \hat{\mathbf{z}}\frac{nI}{2}(\sin\theta_2 - \sin\theta_1) \\
&= \hat{\mathbf{z}}\frac{nI}{2}\left(\frac{l/2 - z'}{\sqrt{(l/2 - z')^2 + a^2}} - \frac{-l/2 - z'}{\sqrt{(-l/2 - z')^2 + a^2}}\right) \\
&= \hat{\mathbf{z}}\frac{nI}{2}\left(\frac{l/2 - z'}{\sqrt{(l/2 - z')^2 + a^2}} + \frac{l/2 + z'}{\sqrt{(l/2 + z')^2 + a^2}}\right) \quad \text{(A/m)}.
\end{aligned}$$

A plot of the magnitude of this function of z' with $a = 5$ cm, $n = 400$ turns$/20$ cm $=$ $20,000$ turns/m, and $I = 12$ A appears in Fig. P5.36.

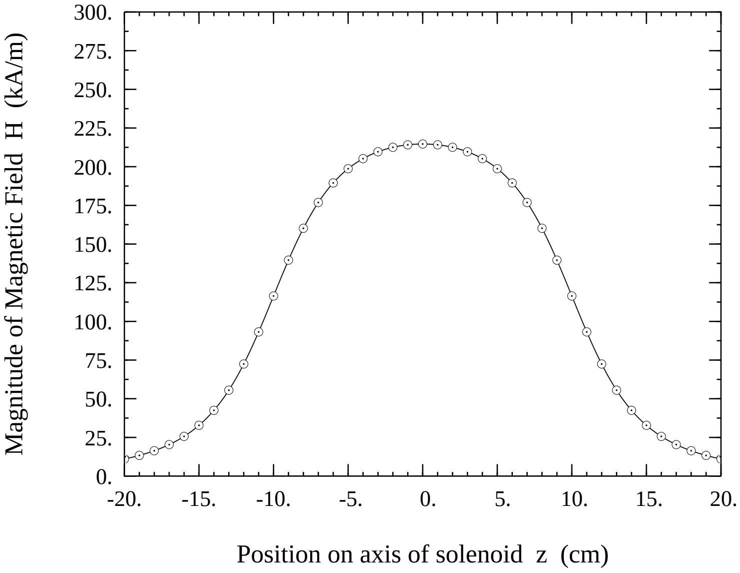


Figure P5.36: Problem 5.36.

Problem 5.37 In terms of the d-c current I, how much magnetic energy is stored in the insulating medium of a 3-m-long, air-filled section of a coaxial transmission line, given that the radius of the inner conductor is 5 cm and the inner radius of the outer conductor is 10 cm?

Solution: From Eq. (5.99), the inductance per unit length of an air-filled coaxial cable is given by

$$L' = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{(H/m)}.$$

Over a length of 2 m, the inductance is

$$L = 2L' = \frac{3 \times 4\pi \times 10^{-7}}{2\pi} \ln\left(\frac{10}{5}\right) = 416 \times 10^{-9} \quad \text{(H)}.$$

From Eq. (5.104), $W_\text{m} = LI^2/2 = 208I^2$ (nJ), where W_m is in nanojoules when I is in amperes. Alternatively, we can use Eq. (5.106) to compute W_m:

$$W_\text{m} = \frac{1}{2} \int_{\mathcal{V}} \mu_0 H^2 \, d\mathcal{V}.$$

From Eq. (5.97), $H = B/\mu_0 = I/2\pi r$, and

$$W_m = \frac{1}{2}\int_{z=0}^{3m}\int_{\phi=0}^{2\pi}\int_{r=a}^{b}\mu_0\left(\frac{I}{2\pi r}\right)^2 r\,dr\,d\phi\,dz = 208I^2 \quad \text{(nJ)}.$$

Problem 5.38 The rectangular loop shown in Fig. 5-48 (P5.38) is coplanar with the long, straight wire carrying the current $I = 20$ A. Determine the magnetic flux through the loop.

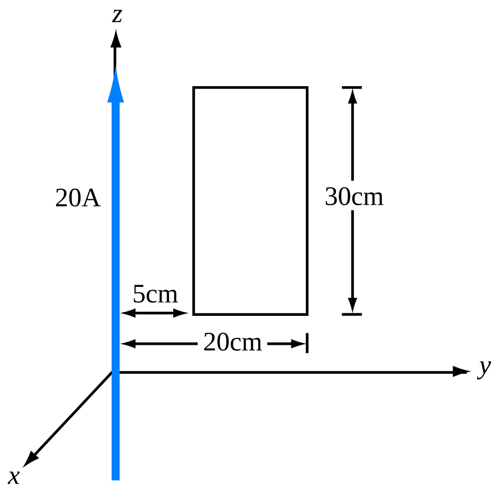


Figure P5.38: Loop and wire arrangement for Problem 5.38.

Solution: The field due to the long wire is, from Eq. (5.30),

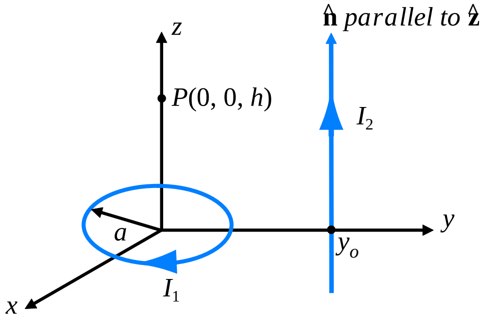
$$\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi r} = -\hat{\mathbf{x}}\,\frac{\mu_0 I}{2\pi r} = -\hat{\mathbf{x}}\,\frac{\mu_0 I}{2\pi y},$$

where in the plane of the loop, $\hat{\boldsymbol{\phi}}$ becomes $-\hat{\mathbf{x}}$ and r becomes y.

The flux through the loop is along $-\hat{\mathbf{x}}$, and the magnitude of the flux is

$$\begin{aligned}\Phi = \int_S \mathbf{B}\cdot d\mathbf{s} &= \frac{\mu_0 I}{2\pi}\int_{5\text{ cm}}^{20\text{ cm}} -\frac{\hat{\mathbf{x}}}{y}\cdot -\hat{\mathbf{x}}\,(30\text{ cm}\times dy)\\ &= \frac{\mu_0 I}{2\pi}\times 0.3\int_{0.05}^{0.2}\frac{dy}{y}\\ &= \frac{0.3\mu_0}{2\pi}\times 20\times \ln\left(\frac{0.2}{0.05}\right) = 1.66\times 10^{-6} \quad \text{(Wb)}.\end{aligned}$$

Problem 5.39 A circular loop of radius a carrying current I_1 is located in the x–y plane as shown in the figure. In addition, an infinitely long wire carrying current I_2 in a direction parallel with the z-axis is located at $y = y_0$.



(a) Determine $\mathbf{H}$ at $P(0,0,h)$.

(b) Evaluate $\mathbf{H}$ for $a = 3$ cm, $y_0 = 10$ cm, $h = 4$ cm, $I_1 = 10$ A, and $I_2 = 20$ A.

Solution:

(a) The magnetic field at $P(0,0,h)$ is composed of $\mathbf{H}_1$ due to the loop and $\mathbf{H}_2$ due to the wire:

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2.$$

From (5.34), with $z = h$,

$$\mathbf{H}_1 = \hat{\mathbf{z}} \frac{I_1 a^2}{2(a^2+h^2)^{3/2}} \quad \text{(A/m)}.$$

From (5.30), the field due to the wire at a distance $r = y_0$ is

$$\mathbf{H}_2 = \hat{\boldsymbol{\phi}} \frac{I_2}{2\pi y_0}$$

where $\hat{\boldsymbol{\phi}}$ is defined with respect to the coordinate system of the wire. Point P is located at an angel $\phi = -90°$ with respect to the wire coordinates. From Table 3-2,

$$\begin{aligned} \hat{\boldsymbol{\phi}} &= -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi \\ &= \hat{\mathbf{x}} \qquad (\text{at } \phi = -90°). \end{aligned}$$

Hence,

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I_1 a^2}{2(a^2+h^2)^{3/2}} + \hat{\mathbf{x}} \frac{I_2}{2\pi y_0} \quad \text{(A/m)}.$$

(b)

$$\mathbf{H} = \hat{\mathbf{z}}36 + \hat{\mathbf{x}}31.83 \quad \text{(A/m)}.$$

Problem 5.40 A cylindrical conductor whose axis is coincident with the z-axis has an internal magnetic field given by

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\,\frac{2}{r}[1-(4r+1)e^{-4r}] \quad \text{(A/m) for } r \leq a$$

where a is the conductor's radius. If $a = 5$ cm, what is the total current flowing in the conductor?

Solution: We can follow either of two possible approaches. The first involves the use of Ampère's law and the second one involves finding **J** from **H** and then **I** from **J**. We will demonstrate both.

Approach 1: Ampère's law

Applying Ampère's law at $r = a$,

$$\begin{aligned}
\oint_C \mathbf{H}\cdot d\boldsymbol{\ell}\big|_{r=a} &= I \\
\int_0^{2\pi} \hat{\boldsymbol{\phi}}\,\frac{2}{r}[1-(4r+1)e^{-4r}]\cdot\hat{\boldsymbol{\phi}}\,r\,d\phi\bigg|_{r=a} &= I \\
I = 4\pi[1-(4a+1)e^{-4a}] \quad \text{(A)}.
\end{aligned}$$

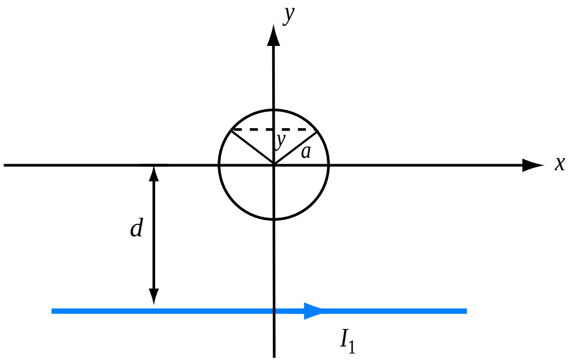
For $a = 5$ cm, $I = 0.22$ (A).

Approach 2: $\mathbf{H} \to \mathbf{J} \to I$

$$\begin{aligned}
\mathbf{J} &= \nabla \times \mathbf{H} \\
&= \hat{\mathbf{z}} \frac{1}{r} \frac{\partial}{\partial r}(rH_\phi) \\
&= \hat{\mathbf{z}} \frac{1}{r} \frac{\partial}{\partial r}(2[1-(4r+1)e^{-4r}]) \\
&= \hat{\mathbf{z}} \frac{1}{r}[-8e^{-4r} + 8(4r+1)e^{-4r}] \\
&= \hat{\mathbf{z}}\, 32e^{-4r}.
\end{aligned}$$

$$\begin{aligned}
I = \int_S \mathbf{J} \cdot d\mathbf{s} &= \int_{r=0}^{a} \hat{\mathbf{z}}\, 32e^{-4r} \cdot \hat{\mathbf{z}}\, 2\pi r\, dr \\
&= 64\pi \int_{r=0}^{a} re^{-4r}\, dr \\
&= \frac{64\pi}{16}[1-(4a+1)e^{-4a}] \\
&= 4\pi[1-(4a+1)e^{-4a}] \quad \text{(A)}.
\end{aligned}$$

Problem 5.41 Determine the mutual inductance between the circular loop and the linear current shown in the figure.



Solution: To calculate the magnetic flux through the loop due to the current in the conductor, we consider a thin strip of thickness dy at location y, as shown. The magnetic field is the same at all points across the strip because they are all equidistant (at $r = d + y$) from the linear conductor. The magnetic flux through the strip is

$$\begin{aligned} d\Phi_{12} &= \mathbf{B}(y) \cdot d\mathbf{s} = \hat{\mathbf{z}} \frac{\mu_0 I}{2\pi(d+y)} \cdot \hat{\mathbf{z}} 2(a^2 - y^2)^{1/2}\, dy \\ &= \frac{\mu_0 I (a^2 - y^2)^{1/2}}{\pi(d+y)}\, dy \\ L_{12} &= \frac{1}{I} \int_S d\Phi_{12} \\ &= \frac{\mu_0}{\pi} \int_{y=-a}^{a} \frac{(a^2 - y^2)^{1/2}\, dy}{(d+y)} \end{aligned}$$

Let $z = d + y \;\rightarrow\; dz = dy$. Hence,

$$\begin{aligned} L_{12} &= \frac{\mu_0}{\pi} \int_{z=d-a}^{d+a} \frac{\sqrt{a^2 - (z-d)^2}}{z}\, dz \\ &= \frac{\mu_0}{\pi} \int_{d-a}^{d+a} \frac{\sqrt{(a^2 - d^2) + 2dz - z^2}}{z}\, dz \\ &= \frac{\mu_0}{\pi} \int \frac{\sqrt{R}}{z}\, dz \end{aligned}$$

where $R = a_0 + b_0 z + c_0 z^2$ and

$$\begin{aligned} a_0 &= a^2 - d^2 \\ b_0 &= 2d \\ c_0 &= -1 \\ \Delta &= 4a_0 c_0 - b_0^2 = -4a^2 < 0 \end{aligned}$$

From Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, 1980, p. 84), we have

$$\int \frac{\sqrt{R}}{z}\, dz = \sqrt{R} + a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}\,.$$

For

$$\sqrt{R}\Big|_{z=d-a}^{d+a} = \sqrt{a^2 - d^2 + 2dz - z^2}\Big|_{z=d-a}^{d+a} = 0 - 0 = 0.$$

For $\displaystyle\int \frac{dz}{z\sqrt{R}}$, several solutions exist depending on the sign of a_0 and Δ.

For this problem, $\Delta < 0$, also let $a_0 < 0$ (i.e., $d > a$). Using the table of integrals,

$$a_0 \int \frac{dz}{z\sqrt{R}} = a_0 \left[\frac{1}{\sqrt{-a_0}} \sin^{-1} \left(\frac{2a_0 + b_0 z}{z\sqrt{b_0^2 - 4a_0 c_0}} \right) \right]_{z=d-a}^{d+a}$$

$$= -\sqrt{d^2 - a^2} \left[\sin^{-1} \left(\frac{a^2 - d^2 + dz}{az} \right) \right]_{z=d-a}^{d+a}$$

$$= -\pi\sqrt{d^2 - a^2}\,.$$

For $\int \frac{dz}{\sqrt{R}}$, different solutions exist depending on the sign of c_0 and Δ.

In this problem, $\Delta < 0$ and $c_0 < 0$. From the table of integrals,

$$\frac{b_0}{z} \int \frac{dz}{\sqrt{R}} = \frac{b_0}{2} \left[\frac{-1}{\sqrt{-c_0}} \sin^{-1} \frac{2c_0 z + b_0}{\sqrt{-\Delta}} \right]_{z=d-a}^{d+a}$$

$$= -d \left[\sin^{-1} \left(\frac{d - z}{a} \right) \right]_{z=d-a}^{d+a} = \pi d.$$

Thus

$$L_{12} = \frac{\mu_0}{\pi} \cdot \left[\pi d - \pi\sqrt{d^2 - a^2} \right]$$

$$= \mu_0 \left[d - \sqrt{d^2 - a^2} \right].$$

Chapter 6: Maxwell's Equations for Time-Varying Fields

Lesson #37

Chapter — Section: 6-1, 6-2

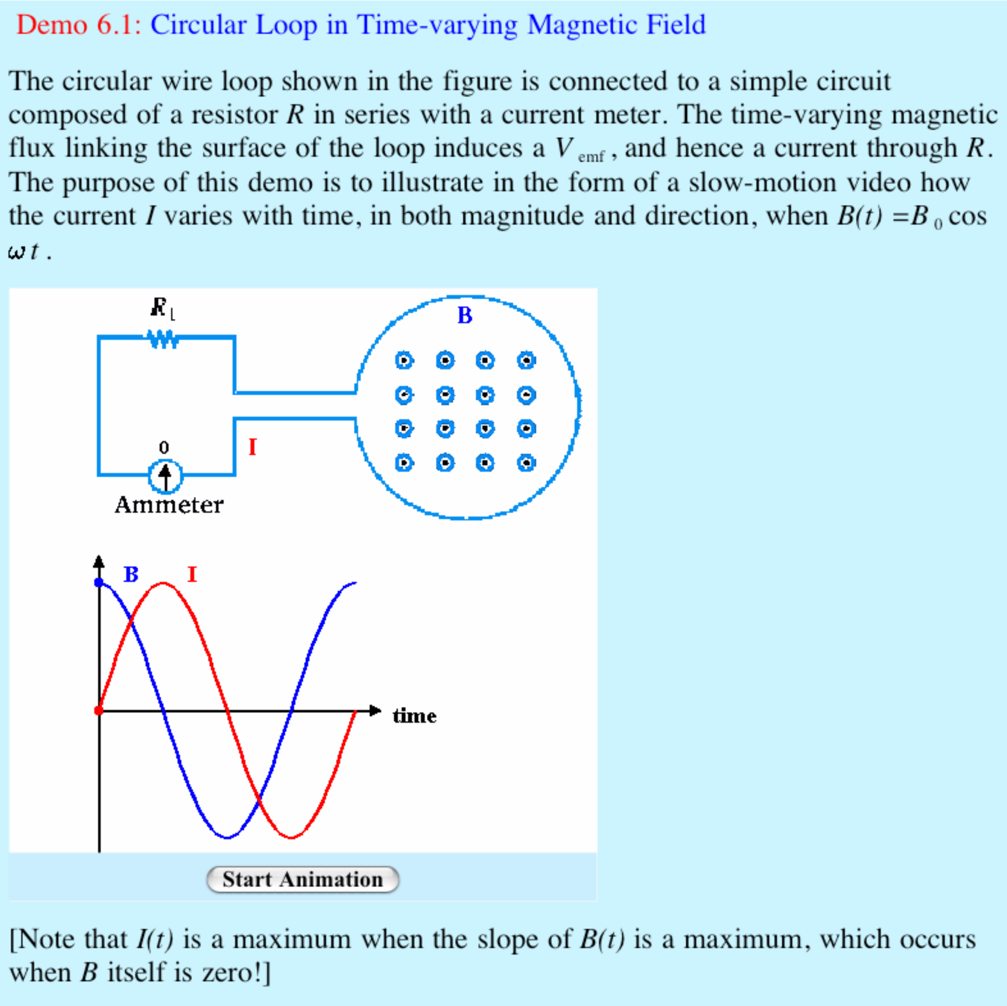
Topics: Faraday's law, stationary loop in changing magnetic field

Highlights:

- Faraday's law
- EMF

Special Illustrations:

- Example 6-1
- Example 6-2
- CD-ROM Demo 6.1
- CD-ROM Modules 6.1 and 6.2



Demo 6.1: Circular Loop in Time-varying Magnetic Field

The circular wire loop shown in the figure is connected to a simple circuit composed of a resistor *R* in series with a current meter. The time-varying magnetic flux linking the surface of the loop induces a V_{emf}, and hence a current through *R*. The purpose of this demo is to illustrate in the form of a slow-motion video how the current *I* varies with time, in both magnitude and direction, when $B(t) = B_0 \cos \omega t$.

[Note that *I(t)* is a maximum when the slope of *B(t)* is a maximum, which occurs when *B* itself is zero!]

Lesson #38

Chapter — Section: 6-3, 6-4

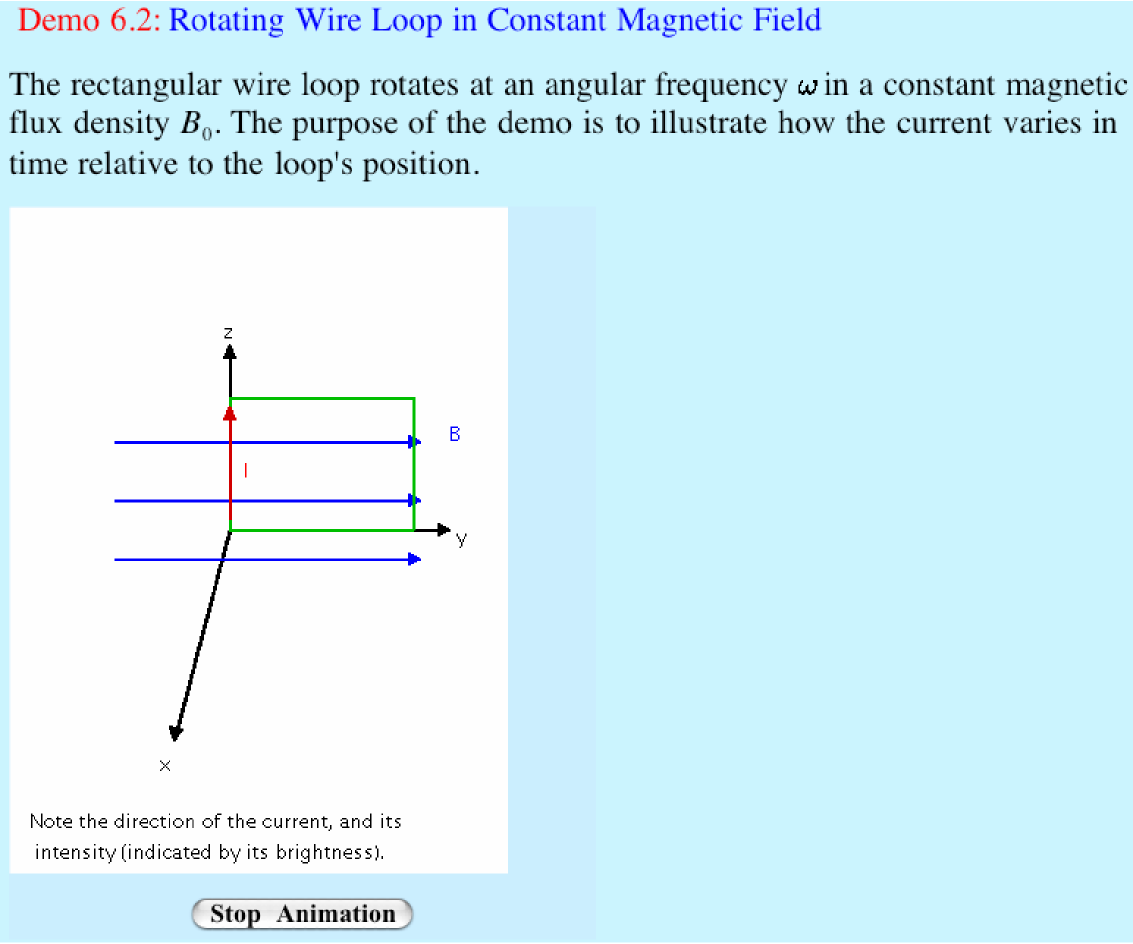
Topics: Ideal transformer, moving conductor

Highlights:

- Transformer voltage and current relations
- EMF for moving conductor

Special Illustrations:

- CD-ROM Modules 6.3 and 6.4
- CD-ROM Demo 6.2



Demo 6.2: Rotating Wire Loop in Constant Magnetic Field

The rectangular wire loop rotates at an angular frequency ω in a constant magnetic flux density B_0. The purpose of the demo is to illustrate how the current varies in time relative to the loop's position.

Note the direction of the current, and its intensity (indicated by its brightness).

Lesson #39

Chapter — Section: 6-5, 6-6

Topics: EM Generator, moving conductor in changing field

Highlights:

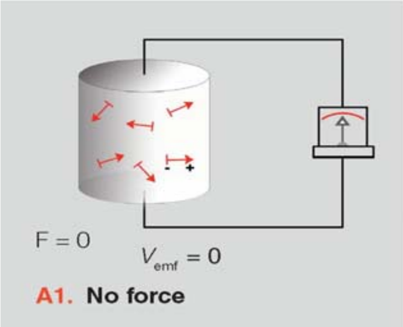
- Motor and generator reciprocity
- EMF for combination of motional and transformer

Special Illustrations:

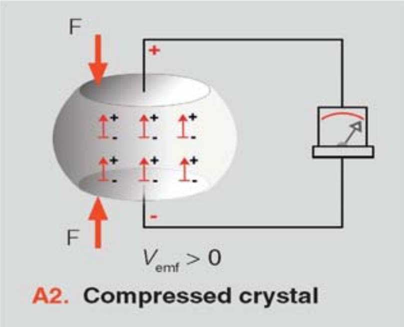
- Technology Brief on “EMF Sensors” (CD-ROM)

EMF Sensors

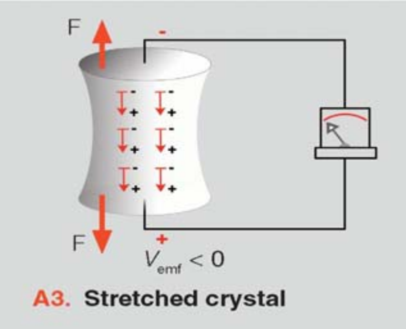
An electromotive force (emf) sensor is a device that can generate an induced voltage in response to an external stimulus. Three types of emf sensors are profiled in this Technical Brief: the piezoelectric transducer, the Faraday magnetic flux sensor, and the thermocouple.



A1. No force



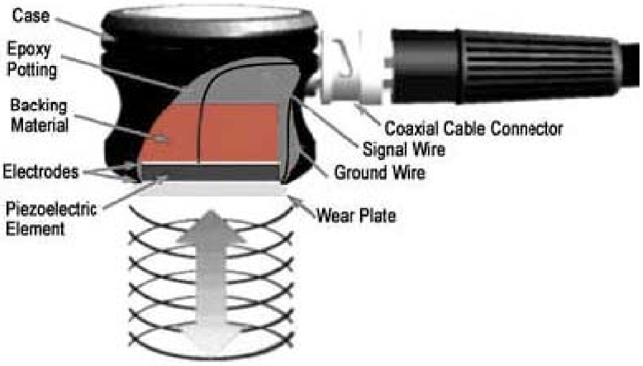
A2. Compressed crystal



A3. Stretched crystal

Piezoelectric Transducers

Piezoelectricity refers to the property of certain crystals, such as quartz, to become electrically polarized when the crystal is subjected to mechanical pressure, thereby exhibiting a voltage across it. The crystal consists of polar domains represented by equivalent dipoles (A). Under the absence of an external force, the polar domains are randomly oriented throughout the material (A1), but when compressive or tensile (stretching) stress is applied to the crystal, the polar domains align themselves along one of the principal axes of the crystal, leading to a net polarization (electric charge) at the crystal surfaces (A2 and A3). Compression and stretching generate voltages of opposite polarity. The piezoelectric effect (*piezein* means to press or squeeze in Greek) was discovered by the Curie brothers, Pierre and Paul-Jacques, in 1880, and a year later Lippmann predicted the converse property, namely that if subjected to an electric field, the crystal would change in shape. Thus, the piezoelectric effect is a reversible (bidirectional) electro-mechanical process.



Lesson #40

Chapter — Section: 6-7, 6-8

Topics: Displacement current, boundary conditions

Highlights:

- Concept of “displacement current”
- Boundary conditions for the dynamic case

Special Illustrations:

- Example 6-7

Lesson #41

Chapter — Section: 6-9, 6-10

Topics: Charge-current continuity, charge dissipation

Highlights:

- Continuity equation
- Relaxation time constant

Special Illustrations:

Lesson #42

Chapter — Section: 6-11

Topics: EM potentials

Highlights:

- Retarded potential
- Relation of potentials to fields in the dynamic case

Special Illustrations:

Example 6-8

Chapter 6

Sections 6-1 to 6-6: Faraday's Law and its Applications

Problem 6.1 The switch in the bottom loop of Fig. 6-17 (P6.1) is closed at $t = 0$ and then opened at a later time t_1. What is the direction of the current I in the top loop (clockwise or counterclockwise) at each of these two times?

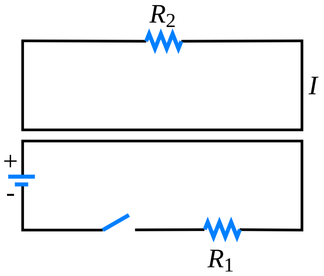


Figure P6.1: Loops of Problem 6.1.

Solution: The magnetic coupling will be strongest at the point where the wires of the two loops come closest. When the switch is closed the current in the bottom loop will start to flow clockwise, which is from left to right in the top portion of the bottom loop. To oppose this change, a current will momentarily flow in the bottom of the top loop from right to left. Thus the current in the top loop is momentarily clockwise when the switch is closed. Similarly, when the switch is opened, the current in the top loop is momentarily counterclockwise.

Problem 6.2 The loop in Fig. 6-18 (P6.2) is in the x–y plane and $\mathbf{B} = \hat{\mathbf{z}}B_0 \sin\omega t$ with B_0 positive. What is the direction of I ($\hat{\boldsymbol{\phi}}$ or $-\hat{\boldsymbol{\phi}}$) at (a) $t = 0$, (b) $\omega t = \pi/4$, and (c) $\omega t = \pi/2$?

Solution: $I = V_{\text{emf}}/R$. Since the single-turn loop is not moving or changing shape with time, $V^{\text{m}}_{\text{emf}} = 0$ V and $V_{\text{emf}} = V^{\text{tr}}_{\text{emf}}$. Therefore, from Eq. (6.8),

$$I = V^{\text{tr}}_{\text{emf}}/R = \frac{-1}{R}\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}.$$

If we take the surface normal to be $+\hat{\mathbf{z}}$, then the right hand rule gives positive flowing current to be in the $+\hat{\boldsymbol{\phi}}$ direction.

$$I = \frac{-A}{R}\frac{\partial}{\partial t}B_0 \sin\omega t = \frac{-AB_0\omega}{R}\cos\omega t \quad \text{(A)},$$

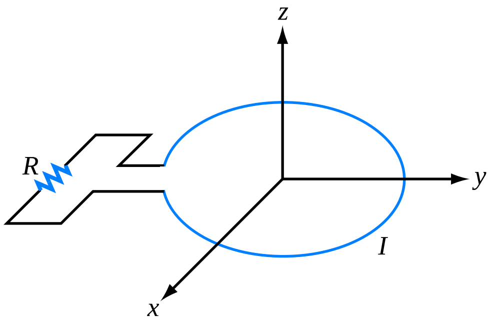


Figure P6.2: Loop of Problem 6.2.

where A is the area of the loop.

(a) A, ω and R are positive quantities. At $t = 0$, $\cos\omega t = 1$ so $I < 0$ and the current is flowing in the $-\hat{\boldsymbol{\phi}}$ direction (so as to produce an induced magnetic field that opposes $\mathbf{B}$).

(b) At $\omega t = \pi/4$, $\cos\omega t = \sqrt{2}/2$ so $I < 0$ and the current is still flowing in the $-\hat{\boldsymbol{\phi}}$ direction.

(c) At $\omega t = \pi/2$, $\cos\omega t = 0$ so $I = 0$. There is no current flowing in either direction.

Problem 6.3 A coil consists of 100 turns of wire wrapped around a square frame of sides 0.25 m. The coil is centered at the origin with each of its sides parallel to the x- or y-axis. Find the induced emf across the open-circuited ends of the coil if the magnetic field is given by

(a) $\mathbf{B} = \hat{\mathbf{z}}\,20e^{-3t}$ (T),

(b) $\mathbf{B} = \hat{\mathbf{z}}\,20\cos x\,\cos 10^3 t$ (T),

(c) $\mathbf{B} = \hat{\mathbf{z}}\,20\cos x\,\sin 2y\,\cos 10^3 t$ (T).

Solution: Since the coil is not moving or changing shape, $V^{\text{m}}_{\text{emf}} = 0$ V and $V_{\text{emf}} = V^{\text{tr}}_{\text{emf}}$. From Eq. (6.6),

$$V_{\text{emf}} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = -N\frac{d}{dt}\int_{-0.125}^{0.125}\int_{-0.125}^{0.125}\mathbf{B}\cdot(\hat{\mathbf{z}}\,dx\,dy),$$

where $N = 100$ and the surface normal was chosen to be in the $+\hat{\mathbf{z}}$ direction.

(a) For $\mathbf{B} = \hat{\mathbf{z}}20e^{-3t}$ (T),

$$V_{\text{emf}} = -100\frac{d}{dt}(20e^{-3t}(0.25)^2) = 375e^{-3t} \quad \text{(V)}.$$

(b) For $\mathbf{B} = \hat{\mathbf{z}}20\cos x \cos 10^3 t$ (T),

$$V_{\text{emf}} = -100\frac{d}{dt}\left(20\cos 10^3 t \int_{x=-0.125}^{0.125}\int_{y=-0.125}^{0.125} \cos x\, dx\, dy\right) = 124.6\sin 10^3 t \quad \text{(kV)}.$$

(c) For $\mathbf{B} = \hat{\mathbf{z}}20\cos x \sin 2y \cos 10^3 t$ (T),

$$V_{\text{emf}} = -100\frac{d}{dt}\left(20\cos 10^3 t \int_{x=-0.125}^{0.125}\int_{y=-0.125}^{0.125} \cos x \sin 2y\, dx\, dy\right) = 0.$$

Problem 6.4 A stationary conducting loop with internal resistance of 0.5 Ω is placed in a time-varying magnetic field. When the loop is closed, a current of 5 A flows through it. What will the current be if the loop is opened to create a small gap and a 2-Ω resistor is connected across its open ends?

Solution: V_{emf} is independent of the resistance which is in the loop. Therefore, when the loop is intact and the internal resistance is only 0.5 Ω,

$$V_{\text{emf}} = 5\text{ A} \times 0.5\ \Omega = 2.5\text{ V}.$$

When the small gap is created, the total resistance in the loop is infinite and the current flow is zero. With a 2-Ω resistor in the gap,

$$I = V_{\text{emf}}/(2\ \Omega + 0.5\ \Omega) = 2.5\text{ V}/2.5\ \Omega = 1 \quad \text{(A)}.$$

Problem 6.5 A circular-loop TV antenna with 0.02 m^2 area is in the presence of a uniform-amplitude 300-MHz signal. When oriented for maximum response, the loop develops an emf with a peak value of 30 (mV). What is the peak magnitude of **B** of the incident wave?

Solution: TV loop antennas have one turn. At maximum orientation, Eq. (6.5) evaluates to $\Phi = \int \mathbf{B} \cdot d\mathbf{s} = \pm BA$ for a loop of area A and a uniform magnetic field with magnitude $B = |\mathbf{B}|$. Since we know the frequency of the field is $f = 300$ MHz, we can express B as $B = B_0 \cos(\omega t + \alpha_0)$ with $\omega = 2\pi \times 300 \times 10^6$ rad/s and α_0 an arbitrary reference phase. From Eq. (6.6),

$$V_{\text{emf}} = -N\frac{d\Phi}{dt} = -A\frac{d}{dt}[B_0\cos(\omega t + \alpha_0)] = AB_0\omega\sin(\omega t + \alpha_0).$$

V_{emf} is maximum when $\sin(\omega t + \alpha_0) = 1$. Hence,

$$30 \times 10^{-3} = AB_0\omega = 0.02 \times B_0 \times 6\pi \times 10^8,$$

which yields $B_0 = 0.8$ (nA/m).

Problem 6.6 The square loop shown in Fig. 6-19 (P6.6) is coplanar with a long, straight wire carrying a current

$$I(t) = 5\cos 2\pi \times 10^4 t \quad \text{(A)}.$$

- **(a)** Determine the emf induced across a small gap created in the loop.
- **(b)** Determine the direction and magnitude of the current that would flow through a 4-Ω resistor connected across the gap. The loop has an internal resistance of 1 Ω.

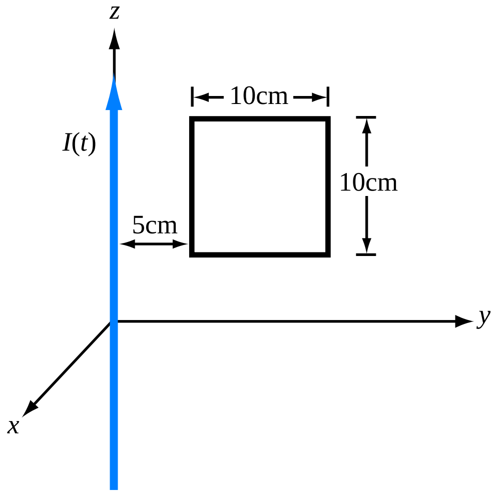


Figure P6.6: Loop coplanar with long wire (Problem 6.6).

Solution:

(a) The magnetic field due to the wire is

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi r} = -\hat{\mathbf{x}}\,\frac{\mu_0 I}{2\pi y}\,,$$

where in the plane of the loop, $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}$ and $r = y$. The flux passing through the loop is

$$\begin{aligned}\Phi = \int_S \mathbf{B}\cdot d\mathbf{s} &= \int_{5\text{ cm}}^{15\text{ cm}} \left(-\hat{\mathbf{x}}\,\frac{\mu_0 I}{2\pi y}\right)\cdot[-\hat{\mathbf{x}}\,10\text{ (cm)}]\,dy\\ &= \frac{\mu_0 I\times 10^{-1}}{2\pi}\ln\frac{15}{5}\\ &= \frac{4\pi\times 10^{-7}\times 5\cos(2\pi\times 10^4 t)\times 10^{-1}}{2\pi}\times 1.1\\ &= 1.1\times 10^{-7}\cos(2\pi\times 10^4 t)\quad\text{(Wb)}.\end{aligned}$$

$$\begin{aligned}V_{\text{emf}} = -\frac{d\Phi}{dt} &= 1.1\times 2\pi\times 10^4\sin(2\pi\times 10^4 t)\times 10^{-7}\\ &= 6.9\times 10^{-3}\sin(2\pi\times 10^4 t)\quad\text{(V)}.\end{aligned}$$

(b)

$$I_{\text{ind}} = \frac{V_{\text{emf}}}{4+1} = \frac{6.9\times 10^{-3}}{5}\sin(2\pi\times 10^4 t) = 1.38\sin(2\pi\times 10^4 t)\quad\text{(mA)}.$$

At $t = 0$, **B** is a maximum, it points in $-\hat{\mathbf{x}}$-direction, and since it varies as $\cos(2\pi\times 10^4 t)$, it is decreasing. Hence, the induced current has to be CCW when looking down on the loop, as shown in the figure.

Problem 6.7 The rectangular conducting loop shown in Fig. 6-20 (P6.7) rotates at 6,000 revolutions per minute in a uniform magnetic flux density given by

$$\mathbf{B} = \hat{\mathbf{y}}\,50\quad\text{(mT)}.$$

Determine the current induced in the loop if its internal resistance is 0.5 Ω.

Solution:

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{S} = \hat{\mathbf{y}}\,50\times 10^{-3}\cdot\hat{\mathbf{y}}(2\times 3\times 10^{-4})\cos\phi(t) = 3\times 10^{-5}\cos\phi(t),$$

$$\phi(t) = \omega t = \frac{2\pi\times 6\times 10^3}{60}\,t = 200\pi t\quad\text{(rad/s)},$$

$$\Phi = 3\times 10^{-5}\cos(200\pi t)\quad\text{(Wb)},$$

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = 3\times 10^{-5}\times 200\pi\sin(200\pi t) = 18.85\times 10^{-3}\sin(200\pi t)\quad\text{(V)},$$

$$I_{\text{ind}} = \frac{V_{\text{emf}}}{0.5} = 37.7\sin(200\pi t)\quad\text{(mA)}.$$

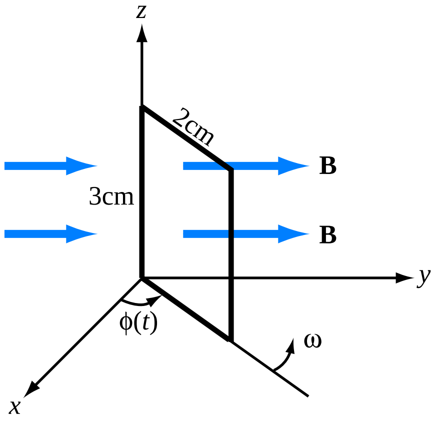


Figure P6.7: Rotating loop in a magnetic field (Problem 6.7).

The direction of the current is CW (if looking at it along $-\hat{\mathbf{x}}$-direction) when the loop is in the first quadrant ($0 \le \phi \le \pi/2$). The current reverses direction in the second quadrant, and reverses again every quadrant.

Problem 6.8 A rectangular conducting loop 5 cm ×10 cm with a small air gap in one of its sides is spinning at 7200 revolutions per minute. If the field **B** is normal to the loop axis and its magnitude is 6×10^{-6} T, what is the peak voltage induced across the air gap?

Solution:

$$\omega = \frac{2\pi \text{ rad/cycle} \times 7200 \text{ cycles/min}}{60 \text{ s/min}} = 240\pi \text{ rad/s},$$

$$A = 5 \text{ cm} \times 10 \text{ cm}/(100 \text{ cm/m})^2 = 5.0 \times 10^{-3} \text{ m}^2.$$

From Eqs. (6.36) or (6.38), $V_{\text{emf}} = A\omega B_0 \sin \omega t$; it can be seen that the peak voltage is

$$V_{\text{emf}}^{\text{peak}} = A\omega B_0 = 5.0 \times 10^{-3} \times 240\pi \times 6 \times 10^{-6} = 22.62 \quad (\mu\text{V}).$$

Problem 6.9 A 50-cm-long metal rod rotates about the z-axis at 90 revolutions per minute, with end 1 fixed at the origin as shown in Fig. 6-21 (P6.9). Determine the induced emf V_{12} if $\mathbf{B} = \hat{\mathbf{z}}\, 2 \times 10^{-4}$ T.

Solution: Since **B** is constant, $V_{\text{emf}} = V_{\text{emf}}^{\text{m}}$. The velocity **u** for any point on the bar is given by $\mathbf{u} = \hat{\boldsymbol{\phi}} r\omega$, where

$$\omega = \frac{2\pi \text{ rad/cycle} \times (90 \text{ cycles/min})}{(60 \text{ s/min})} = 3\pi \text{ rad/s}.$$

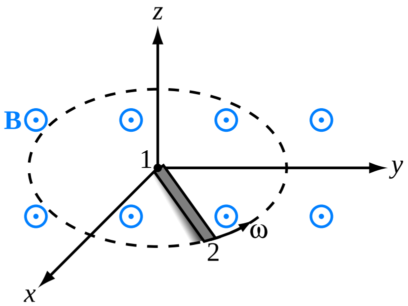


Figure P6.9: Rotating rod of Problem 6.9.

From Eq. (6.24),

$$
\begin{aligned}
V_{12} = V_{\text{emf}}^{\text{m}} = \int_{2}^{1} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} &= \int_{r=0.5}^{0} (\hat{\boldsymbol{\phi}} 3\pi r \times \hat{\mathbf{z}} 2 \times 10^{-4}) \cdot \hat{\mathbf{r}}\, dr \\
&= 6\pi \times 10^{-4} \int_{r=0.5}^{0} r\, dr \\
&= 3\pi \times 10^{-4} r^2 \Big|_{0.5}^{0} \\
&= -3\pi \times 10^{-4} \times 0.25 = -236 \quad (\mu\text{V}).
\end{aligned}
$$

Problem 6.10 The loop shown in Fig. 6-22 (P6.10) moves away from a wire carrying a current $I_1 = 10$ (A) at a constant velocity $\mathbf{u} = \hat{\mathbf{y}} 7.5$ (m/s). If $R = 10\ \Omega$ and the direction of I_2 is as defined in the figure, find I_2 as a function of y_0, the distance between the wire and the loop. Ignore the internal resistance of the loop.

Solution: Assume that the wire carrying current I_1 is in the same plane as the loop. The two identical resistors are in series, so $I_2 = V_{\text{emf}}/2R$, where the induced voltage is due to motion of the loop and is given by Eq. (6.26):

$$V_{\text{emf}} = V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

The magnetic field $\mathbf{B}$ is created by the wire carrying I_1. Choosing $\hat{\mathbf{z}}$ to coincide with the direction of I_1, Eq. (5.30) gives the external magnetic field of a long wire to be

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I_1}{2\pi r}.$$

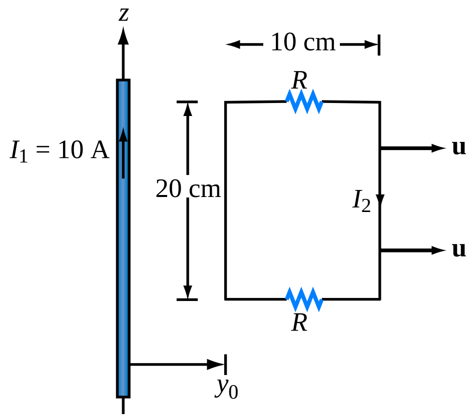


Figure P6.10: Moving loop of Problem 6.10.

For positive values of y_0 in the y-z plane, $\hat{\mathbf{y}} = \hat{\mathbf{r}}$, so

$$\mathbf{u} \times \mathbf{B} = \hat{\mathbf{y}}|\mathbf{u}| \times \mathbf{B} = \hat{\mathbf{r}}|\mathbf{u}| \times \hat{\boldsymbol{\phi}}\frac{\mu_0 I_1}{2\pi r} = \hat{\mathbf{z}}\frac{\mu_0 I_1 u}{2\pi r}\,.$$

Integrating around the four sides of the loop with $d\mathbf{l} = \hat{\mathbf{z}}\,dz$ and the limits of integration chosen in accordance with the assumed direction of I_2, and recognizing that only the two sides without the resistors contribute to $V^{\text{m}}_{\text{emf}}$, we have

$$\begin{aligned} V^{\text{m}}_{\text{emf}} &= \int_0^{0.2} \left(\hat{\mathbf{z}}\frac{\mu_0 I_1 u}{2\pi r}\right)\bigg|_{r=y_0} \cdot (\hat{\mathbf{z}}\,dz) + \int_{0.2}^{0} \left(\hat{\mathbf{z}}\frac{\mu_0 I_1 u}{2\pi r}\right)\bigg|_{r=y_0+0.1} \cdot (\hat{\mathbf{z}}\,dz) \\ &= \frac{4\pi \times 10^{-7} \times 10 \times 7.5 \times 0.2}{2\pi}\left(\frac{1}{y_0} - \frac{1}{y_0+0.1}\right) \\ &= 3 \times 10^{-6}\left(\frac{1}{y_0} - \frac{1}{y_0+0.1}\right) \quad \text{(V)}, \end{aligned}$$

and therefore

$$I_2 = \frac{V^{\text{m}}_{\text{emf}}}{2R} = 150\left(\frac{1}{y_0} - \frac{1}{y_0+0.1}\right) \quad \text{(nA)}.$$

Problem 6.11 The conducting cylinder shown in Fig. 6-23 (P6.11) rotates about its axis at 1,200 revolutions per minute in a radial field given by

$$\mathbf{B} = \hat{\mathbf{r}}\,6 \quad \text{(T)}.$$

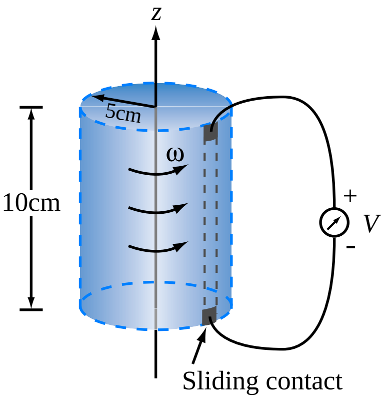


Figure P6.11: Rotating cylinder in a magnetic field (Problem 6.11).

The cylinder, whose radius is 5 cm and height 10 cm, has sliding contacts at its top and bottom connected to a voltmeter. Determine the induced voltage.

Solution: The surface of the cylinder has velocity $\mathbf{u}$ given by

$$\mathbf{u} = \hat{\boldsymbol{\phi}}\omega r = \hat{\boldsymbol{\phi}}2\pi \times \frac{1,200}{60} \times 5 \times 10^{-2} = \hat{\boldsymbol{\phi}}2\pi \quad \text{(m/s)},$$

$$V_{12} = \int_0^L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_0^{0.1} (\hat{\boldsymbol{\phi}}2\pi \times \hat{\mathbf{r}}6) \cdot \hat{\mathbf{z}}\, dz = -3.77 \quad \text{(V)}.$$

Problem 6.12 The electromagnetic generator shown in Fig. 6-12 is connected to an electric bulb with a resistance of 150 Ω. If the loop area is 0.1 m^2 and it rotates at 3,600 revolutions per minute in a uniform magnetic flux density $B_0 = 0.4$ T, determine the amplitude of the current generated in the light bulb.

Solution: From Eq. (6.38), the sinusoidal voltage generated by the a-c generator is $V_{\text{emf}} = A\omega B_0 \sin(\omega t + C_0) = V_0 \sin(\omega t + C_0)$. Hence,

$$V_0 = A\omega B_0 = 0.1 \times \frac{2\pi \times 3,600}{60} \times 0.4 = 15.08 \quad \text{(V)},$$

$$I = \frac{V_0}{R} = \frac{15.08}{150} = 0.1 \quad \text{(A)}.$$

Problem 6.13 The circular disk shown in Fig. 6-24 (P6.13) lies in the x–y plane and rotates with uniform angular velocity ω about the z-axis. The disk is of radius a and is present in a uniform magnetic flux density $\mathbf{B} = \hat{\mathbf{z}}B_0$. Obtain an expression for the emf induced at the rim relative to the center of the disk.

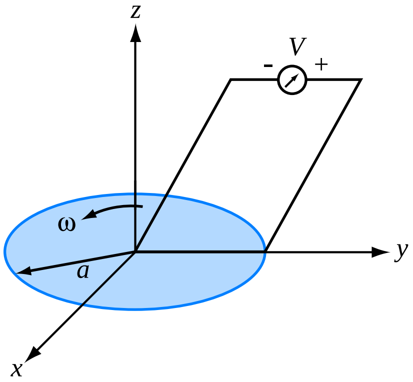


Figure P6.13: Rotating circular disk in a magnetic field (Problem 6.13).

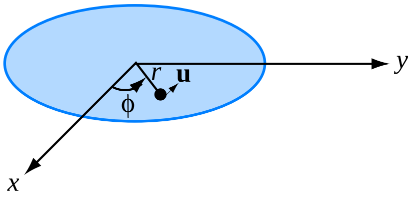


Figure P6.13: (a) Velocity vector **u**.

Solution: At a radial distance r, the velocity is

$$\mathbf{u} = \hat{\boldsymbol{\phi}}\,\omega r$$

where ϕ is the angle in the x–y plane shown in the figure. The induced voltage is

$$V = \int_0^a (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_0^a [(\hat{\boldsymbol{\phi}}\,\omega r) \times \hat{\mathbf{z}} B_0] \cdot \hat{\mathbf{r}}\; dr.$$

$\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}$ is along $\hat{\mathbf{r}}$. Hence,

$$V = \omega B_0 \int_0^a r\,dr = \frac{\omega B_0 a^2}{2}.$$

Section 6-7: Displacement Current

Problem 6.14 The plates of a parallel-plate capacitor have areas 10 cm^2 each and are separated by 2 cm. The capacitor is filled with a dielectric material with

$\varepsilon = 4\varepsilon_0$, and the voltage across it is given by $V(t) = 30\cos 2\pi \times 10^6 t$ (V). Find the displacement current.

Solution: Since the voltage is of the form given by Eq. (6.46) with $V_0 = 30$ V and $\omega = 2\pi \times 10^6$ rad/s, the displacement current is given by Eq. (6.49):

$$\begin{aligned} I_d &= -\frac{\varepsilon A}{d} V_0 \omega \sin \omega t \\ &= -\frac{4 \times 8.854 \times 10^{-12} \times 10 \times 10^{-4}}{2 \times 10^{-2}} \times 30 \times 2\pi \times 10^6 \sin(2\pi \times 10^6 t) \\ &= -0.33 \sin(2\pi \times 10^6 t) \quad \text{(mA)}. \end{aligned}$$

Problem 6.15 A coaxial capacitor of length $l = 6$ cm uses an insulating dielectric material with $\varepsilon_r = 9$. The radii of the cylindrical conductors are 0.5 cm and 1 cm. If the voltage applied across the capacitor is

$$V(t) = 50 \sin(120\pi t) \quad \text{(V)},$$

what is the displacement current?

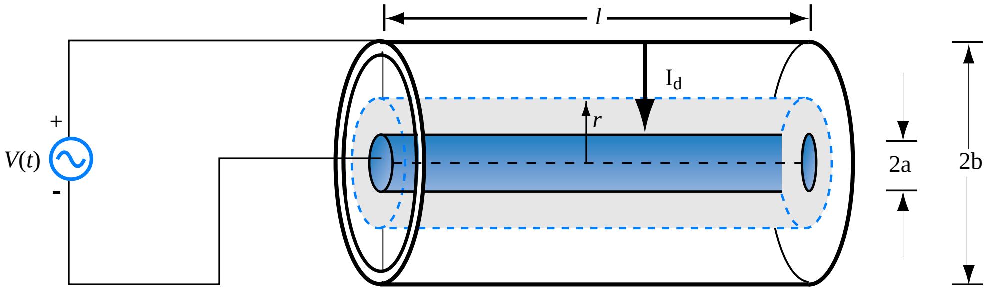


Figure P6.15:

Solution: To find the displacement current, we need to know **E** in the dielectric space between the cylindrical conductors. From Eqs. (4.114) and (4.115),

$$\mathbf{E} = -\hat{\mathbf{r}} \frac{Q}{2\pi\varepsilon r l},$$
$$V = \frac{Q}{2\pi\varepsilon l} \ln\left(\frac{b}{a}\right).$$

Hence,

$$\mathbf{E} = -\hat{\mathbf{r}} \frac{V}{r \ln\left(\frac{b}{a}\right)} = -\hat{\mathbf{r}} \frac{50 \sin(120\pi t)}{r \ln 2} = -\hat{\mathbf{r}} \frac{72.1}{r} \sin(120\pi t) \quad \text{(V/m)},$$

$$\begin{aligned}\mathbf{D} &= \varepsilon\mathbf{E}\\ &= \varepsilon_{\mathrm{r}}\varepsilon_0\mathbf{E}\\ &= -\hat{\mathbf{r}}\,9\times 8.85\times 10^{-12}\times\frac{72.1}{r}\sin(120\pi t)\\ &= -\hat{\mathbf{r}}\,\frac{5.75\times 10^{-9}}{r}\sin(120\pi t)\quad (\mathrm{C/m^2}).\end{aligned}$$

The displacement current flows between the conductors through an imaginary cylindrical surface of length l and radius r. The current flowing from the outer conductor to the inner conductor along $-\hat{\mathbf{r}}$ crosses surface $\mathbf{S}$ where

$$\mathbf{S} = -\hat{\mathbf{r}}\,2\pi rl.$$

Hence,

$$\begin{aligned}I_{\mathrm{d}} = \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{S} &= -\hat{\mathbf{r}}\,\frac{\partial}{\partial t}\left(\frac{5.75\times 10^{-9}}{r}\sin(120\pi t)\right)\cdot(-\hat{\mathbf{r}}\,2\pi rl)\\ &= 5.75\times 10^{-9}\times 120\pi\times 2\pi l\cos(120\pi t)\\ &= 0.82\cos(120\pi t)\quad (\mu\mathrm{A}).\end{aligned}$$

Alternatively, since the coaxial capacitor is lossless, its displacement current has to be equal to the conduction current flowing through the wires connected to the voltage sources. The capacitance of a coaxial capacitor is given by (4.116) as

$$C = \frac{2\pi\varepsilon l}{\ln\left(\frac{b}{a}\right)}\,.$$

The current is

$$I = C\,\frac{dV}{dt} = \frac{2\pi\varepsilon l}{\ln\left(\frac{b}{a}\right)}\,[120\pi\times 50\cos(120\pi t)] = 0.82\cos(120\pi t)\quad (\mu\mathrm{A}),$$

which is the same answer we obtained before.

Problem 6.16 The parallel-plate capacitor shown in Fig. 6-25 (P6.16) is filled with a lossy dielectric material of relative permittivity ε_{r} and conductivity σ. The separation between the plates is d and each plate is of area A. The capacitor is connected to a time-varying voltage source $V(t)$.

(a) Obtain an expression for I_{c}, the conduction current flowing between the plates inside the capacitor, in terms of the given quantities.

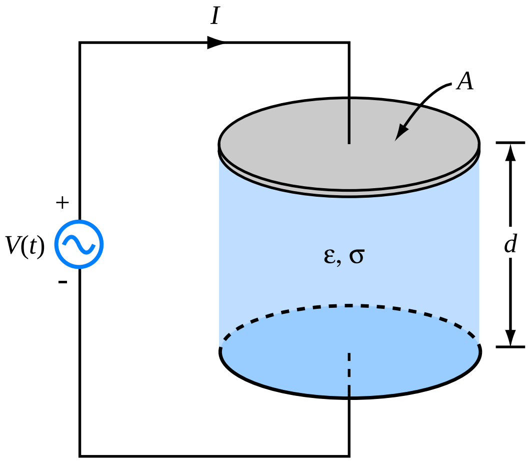


Figure P6.16: Parallel-plate capacitor containing a lossy dielectric material (Problem 6.16).

(b) Obtain an expression for I_d, the displacement current flowing inside the capacitor.

(c) Based on your expression for parts (a) and (b), give an equivalent-circuit representation for the capacitor.

(d) Evaluate the values of the circuit elements for $A = 4$ cm^2, $d = 0.5$ cm, $\varepsilon_r = 4$, $\sigma = 2.5$ (S/m), and $V(t) = 10\cos(3\pi \times 10^3 t)$ (V).

Solution:

(a)

$$R = \frac{d}{\sigma A}, \qquad I_\text{c} = \frac{V}{R} = \frac{V\sigma A}{d}.$$

(b)

$$E = \frac{V}{d}, \qquad I_\text{d} = \frac{\partial D}{\partial t} \cdot A = \varepsilon A \frac{\partial E}{\partial t} = \frac{\varepsilon A}{d} \frac{\partial V}{\partial t}.$$

(c) The conduction current is directly proportional to V, as characteristic of a resistor, whereas the displacement current varies as $\partial V/\partial t$, which is characteristic of a capacitor. Hence,

$$R = \frac{d}{\sigma A} \qquad \text{and} \qquad C = \frac{\varepsilon A}{d}.$$

(d)

$$R = \frac{0.5 \times 10^{-2}}{2.5 \times 4 \times 10^{-4}} = 5\ \Omega,$$

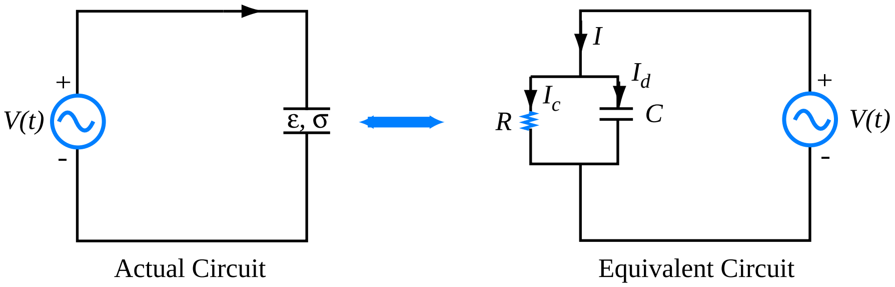


Figure P6.16: (a) Equivalent circuit.

$$C = \frac{4 \times 8.85 \times 10^{-12} \times 4 \times 10^{-4}}{0.5 \times 10^{-2}} = 2.84 \times 10^{-12} \text{ F.}$$

Problem 6.17 An electromagnetic wave propagating in seawater has an electric field with a time variation given by $\mathbf{E} = \hat{\mathbf{z}} E_0 \cos \omega t$. If the permittivity of water is $81\varepsilon_0$ and its conductivity is 4 (S/m), find the ratio of the magnitudes of the conduction current density to displacement current density at each of the following frequencies: (a) 1 kHz, (b) 1 MHz, (c) 1 GHz, (d) 100 GHz.

Solution: From Eq. (6.44), the displacement current density is given by

$$\mathbf{J}_\text{d} = \frac{\partial}{\partial t}\mathbf{D} = \varepsilon \frac{\partial}{\partial t}\mathbf{E}$$

and, from Eq. (4.67), the conduction current is $\mathbf{J} = \sigma \mathbf{E}$. Converting to phasors and taking the ratio of the magnitudes,

$$\left| \frac{\widetilde{\mathbf{J}}}{\widetilde{\mathbf{J}}_\text{d}} \right| = \left| \frac{\sigma \widetilde{\mathbf{E}}}{j\omega\varepsilon_r\varepsilon_0 \widetilde{\mathbf{E}}} \right| = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0}.$$

(a) At $f = 1$ kHz, $\omega = 2\pi \times 10^3$ rad/s, and

$$\left| \frac{\widetilde{\mathbf{J}}}{\widetilde{\mathbf{J}}_\text{d}} \right| = \frac{4}{2\pi \times 10^3 \times 81 \times 8.854 \times 10^{-12}} = 888 \times 10^3.$$

The displacement current is negligible.

(b) At $f = 1$ MHz, $\omega = 2\pi \times 10^6$ rad/s, and

$$\left| \frac{\widetilde{\mathbf{J}}}{\widetilde{\mathbf{J}}_\text{d}} \right| = \frac{4}{2\pi \times 10^6 \times 81 \times 8.854 \times 10^{-12}} = 888.$$

The displacement current is practically negligible.

(c) At $f = 1$ GHz, $\omega = 2\pi \times 10^9$ rad/s, and

$$\left| \frac{\widetilde{\mathbf{J}}}{\widetilde{\mathbf{J}}_{\mathrm{d}}} \right| = \frac{4}{2\pi \times 10^9 \times 81 \times 8.854 \times 10^{-12}} = 0.888.$$

Neither the displacement current nor the conduction current are negligible.

(d) At $f = 100$ GHz, $\omega = 2\pi \times 10^{11}$ rad/s, and

$$\left| \frac{\widetilde{\mathbf{J}}}{\widetilde{\mathbf{J}}_{\mathrm{d}}} \right| = \frac{4}{2\pi \times 10^{11} \times 81 \times 8.854 \times 10^{-12}} = 8.88 \times 10^{-3}.$$

The conduction current is practically negligible.

Sections 6-9 and 6-10: Continuity Equation and Charge Dissipation

Problem 6.18 At $t = 0$, charge density ρ_{v0} was introduced into the interior of a material with a relative permittivity $\varepsilon_{\mathrm{r}} = 9$. If at $t = 1$ μs the charge density has dissipated down to $10^{-3}\rho_{\mathrm{v0}}$, what is the conductivity of the material?

Solution: We start by using Eq. (6.61) to find τ_{r}:

$$\rho_{\mathrm{v}}(t) = \rho_{\mathrm{v0}} e^{-t/\tau_{\mathrm{r}}},$$

or

$$10^{-3}\rho_{\mathrm{v0}} = \rho_{\mathrm{v0}} e^{-10^{-6}/\tau_{\mathrm{r}}},$$

which gives

$$\ln 10^{-3} = -\frac{10^{-6}}{\tau_{\mathrm{r}}},$$

or

$$\tau_{\mathrm{r}} = -\frac{10^{-6}}{\ln 10^{-3}} = 1.45 \times 10^{-7} \quad \text{(s)}.$$

But $\tau_{\mathrm{r}} = \varepsilon/\sigma = 9\varepsilon_0/\sigma$. Hence

$$\sigma = \frac{9\varepsilon_0}{\tau_{\mathrm{r}}} = \frac{9 \times 8.854 \times 10^{-12}}{1.45 \times 10^{-7}} = 5.5 \times 10^{-4} \quad \text{(S/m)}.$$

Problem 6.19 If the current density in a conducting medium is given by

$$\mathbf{J}(x,y,z;t) = (\hat{\mathbf{x}}z^2 - \hat{\mathbf{y}}4y^2 + \hat{\mathbf{z}}2x)\cos\omega t,$$

determine the corresponding charge distribution $\rho_\mathrm{v}(x,y,z;t)$.

Solution: Eq. (6.58) is given by

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_\mathrm{v}}{\partial t}. \tag{24}$$

The divergence of **J** is

$$\begin{aligned}\nabla\cdot\mathbf{J} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot(\hat{\mathbf{x}}z^2 - \hat{\mathbf{y}}4y^2 + \hat{\mathbf{z}}2x)\cos\omega t \\ &= -4\frac{\partial}{\partial y}(y^2\cos\omega t) = -8y\cos\omega t.\end{aligned}$$

Using this result in Eq. (24) and then integrating both sides with respect to t gives

$$\rho_\mathrm{v} = -\int(\nabla\cdot\mathbf{J})\,dt = -\int -8y\cos\omega t\,dt = \frac{8y}{\omega}\sin\omega t + C_0,$$

where C_0 is a constant of integration.

Problem 6.20 In a certain medium, the direction of current density **J** points in the radial direction in cylindrical coordinates and its magnitude is independent of both ϕ and z. Determine **J**, given that the charge density in the medium is

$$\rho_\mathrm{v} = \rho_0 r\cos\omega t \quad (\mathrm{C/m^3}).$$

Solution: Based on the given information,

$$\mathbf{J} = \hat{\mathbf{r}}J_\mathrm{r}(r).$$

With $J_\phi = J_z = 0$, in cylindrical coordinates the divergence is given by

$$\nabla\cdot\mathbf{J} = \frac{1}{r}\frac{\partial}{\partial r}(rJ_\mathrm{r}).$$

From Eq. (6.54),

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_\mathrm{v}}{\partial t} = -\frac{\partial}{\partial t}(\rho_0 r\cos\omega t) = \rho_0 r\omega\sin\omega t.$$

Hence

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(rJ_{\rm r}) &= \rho_0 r\omega\sin\omega t,\\
\frac{\partial}{\partial r}(rJ_{\rm r}) &= \rho_0 r^2\omega\sin\omega t,\\
\int_0^r \frac{\partial}{\partial r}(rJ_{\rm r})\,dr &= \rho_0\omega\sin\omega t\int_0^r r^2\,dr,\\
rJ_{\rm r}\big|_0^r &= (\rho_0\omega\sin\omega t)\left.\frac{r^3}{3}\right|_0^r,\\
J_{\rm r} &= \frac{\rho_0\omega r^2}{3}\sin\omega t,
\end{aligned}$$

and

$$\mathbf{J} = \hat{\mathbf{r}}J_{\rm r} = \hat{\mathbf{r}}\frac{\rho_0\omega r^2}{3}\sin\omega t \quad (\text{A/m}^2).$$

Problem 6.21 If we were to characterize how good a material is as an insulator by its resistance to dissipating charge, which of the following two materials is the better insulator?

Dry Soil: $\varepsilon_{\rm r} = 2.5$, $\sigma = 10^{-4}$ (S/m)
Fresh Water: $\varepsilon_{\rm r} = 80$, $\sigma = 10^{-3}$ (S/m)

Solution: Relaxation time constant $\tau_{\rm r} = \frac{\varepsilon}{\sigma}$.

$$\text{For dry soil,}\quad \tau_{\rm r} = \frac{2.5}{10^{-4}} = 2.5\times10^4 \text{ s.}$$
$$\text{For fresh water,}\quad \tau_{\rm r} = \frac{80}{10^{-3}} = 8\times10^4 \text{ s.}$$

Since it takes longer for charge to dissipate in fresh water, it is a better insulator than dry soil.

Sections 6-11: Electromagnetic Potentials

Problem 6.22 The electric field of an electromagnetic wave propagating in air is given by

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}4\cos(6\times10^8 t - 2z) + \hat{\mathbf{y}}3\sin(6\times10^8 t - 2z) \quad (\text{V/m}).$$

Find the associated magnetic field $\mathbf{H}(z,t)$.

Solution: Converting to phasor form, the electric field is given by

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}4e^{-j2z} - j\hat{\mathbf{y}}3e^{-j2z} \quad \text{(V/m)},$$

which can be used with Eq. (6.87) to find the magnetic field:

$$\begin{aligned}
\widetilde{\mathbf{H}}(z) &= \frac{1}{-j\omega\mu}\nabla\times\widetilde{\mathbf{E}} \\
&= \frac{1}{-j\omega\mu}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4e^{-j2z} & -j3e^{-j2z} & 0 \end{vmatrix} \\
&= \frac{1}{-j\omega\mu}(\hat{\mathbf{x}}6e^{-j2z} - \hat{\mathbf{y}}\,j8e^{-j2z}) \\
&= \frac{j}{6\times10^8\times4\pi\times10^{-7}}(\hat{\mathbf{x}}6 - \hat{\mathbf{y}}\,j8)e^{-j2z} = j\hat{\mathbf{x}}8.0e^{-j2z} + \hat{\mathbf{y}}10.6e^{-j2z} \quad \text{(mA/m)}.
\end{aligned}$$

Converting back to instantaneous values, this is

$$\mathbf{H}(t,z) = -\hat{\mathbf{x}}8.0\sin(6\times10^8t - 2z) + \hat{\mathbf{y}}10.6\cos(6\times10^8t - 2z) \quad \text{(mA/m)}.$$

Problem 6.23 The magnetic field in a dielectric material with $\varepsilon = 4\varepsilon_0$, $\mu = \mu_0$, and $\sigma = 0$ is given by

$$\mathbf{H}(y,t) = \hat{\mathbf{x}}5\cos(2\pi\times10^7t + ky) \quad \text{(A/m)}.$$

Find k and the associated electric field $\mathbf{E}$.

Solution: In phasor form, the magnetic field is given by $\widetilde{\mathbf{H}} = \hat{\mathbf{x}}5e^{jky}$ (A/m). From Eq. (6.86),

$$\widetilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\nabla\times\widetilde{\mathbf{H}} = \frac{-jk}{j\omega\varepsilon}\hat{\mathbf{z}}5e^{jky}$$

and, from Eq. (6.87),

$$\widetilde{\mathbf{H}} = \frac{1}{-j\omega\mu}\nabla\times\widetilde{\mathbf{E}} = \frac{-jk^2}{-j\omega^2\varepsilon\mu}\hat{\mathbf{x}}5e^{jky},$$

which, together with the original phasor expression for $\widetilde{\mathbf{H}}$, implies that

$$k = \omega\sqrt{\varepsilon\mu} = \frac{\omega\sqrt{\varepsilon_r}}{c} = \frac{2\pi\times10^7\sqrt{4}}{3\times10^8} = \frac{4\pi}{30} \quad \text{(rad/m)}.$$

Inserting this value in the expression for $\widetilde{\mathbf{E}}$ above,

$$\widetilde{\mathbf{E}} = -\hat{\mathbf{z}}\frac{4\pi/30}{2\pi\times10^7\times4\times8.854\times10^{-12}}5e^{j4\pi y/30} = -\hat{\mathbf{z}}941e^{j4\pi y/30} \quad \text{(V/m)}.$$

Problem 6.24 Given an electric field

$$\mathbf{E} = \hat{\mathbf{x}}E_0\sin ay\cos(\omega t - kz),$$

where E_0, a, ω, and k are constants, find $\mathbf{H}$.

Solution:

$$\begin{aligned}
\mathbf{E} &= \hat{\mathbf{x}}E_0\sin ay\cos(\omega t - kz),\\
\widetilde{\mathbf{E}} &= \hat{\mathbf{x}}E_0\sin ay\, e^{-jkz},\\
\widetilde{\mathbf{H}} &= -\frac{1}{j\omega\mu}\nabla\times\widetilde{\mathbf{E}}\\
&= -\frac{1}{j\omega\mu}\left[\hat{\mathbf{y}}\frac{\partial}{\partial z}(E_0\sin ay\, e^{-jkz}) - \hat{\mathbf{z}}\frac{\partial}{\partial y}(E_0\sin ay\, e^{-jkz})\right]\\
&= \frac{E_0}{\omega\mu}[\hat{\mathbf{y}}k\sin ay - \hat{\mathbf{z}}\, ja\cos ay]e^{-jkz},\\
\mathbf{H} &= \mathfrak{Re}[\widetilde{\mathbf{H}}e^{j\omega t}]\\
&= \mathfrak{Re}\left\{\frac{E_0}{\omega\mu}[\hat{\mathbf{y}}k\sin ay + \hat{\mathbf{z}}a\cos ay\, e^{-j\pi/2}]e^{-jkz}e^{j\omega t}\right\}\\
&= \frac{E_0}{\omega\mu}\left[\hat{\mathbf{y}}k\sin ay\cos(\omega t - kz) + \hat{\mathbf{z}}a\cos ay\cos\left(\omega t - kz - \frac{\pi}{2}\right)\right]\\
&= \frac{E_0}{\omega\mu}[\hat{\mathbf{y}}k\sin ay\cos(\omega t - kz) + \hat{\mathbf{z}}a\cos ay\sin(\omega t - kz)]\,.
\end{aligned}$$

Problem 6.25 The electric field radiated by a short dipole antenna is given in spherical coordinates by

$$\mathbf{E}(R,\theta;t) = \hat{\boldsymbol{\theta}}\frac{2\times10^{-2}}{R}\sin\theta\,\cos(6\pi\times10^8t - 2\pi R) \quad \text{(V/m)}.$$

Find $\mathbf{H}(R,\theta;t)$.

Solution: Converting to phasor form, the electric field is given by

$$\widetilde{\mathbf{E}}(R,\theta) = \hat{\boldsymbol{\theta}}E_\theta = \hat{\boldsymbol{\theta}}\frac{2\times10^{-2}}{R}\sin\theta e^{-j2\pi R} \quad \text{(V/m)},$$

which can be used with Eq. (6.87) to find the magnetic field:

$$\begin{aligned}\widetilde{\mathbf{H}}(R,\theta) &= \frac{1}{-j\omega\mu}\nabla\times\widetilde{\mathbf{E}} = \frac{1}{-j\omega\mu}\left[\hat{\mathbf{R}}\frac{1}{R\sin\theta}\frac{\partial E_\theta}{\partial\phi} + \hat{\boldsymbol{\phi}}\frac{1}{R}\frac{\partial}{\partial R}(RE_\theta)\right] \\ &= \frac{1}{-j\omega\mu}\hat{\boldsymbol{\phi}}\frac{2\times10^{-2}}{R}\sin\theta\,\frac{\partial}{\partial R}(e^{-j2\pi R}) \\ &= \hat{\boldsymbol{\phi}}\frac{2\pi}{6\pi\times10^{8}\times4\pi\times10^{-7}}\frac{2\times10^{-2}}{R}\sin\theta e^{-j2\pi R} \\ &= \hat{\boldsymbol{\phi}}\frac{53}{R}\sin\theta\,e^{-j2\pi R} \quad (\mu\text{A/m}).\end{aligned}$$

Converting back to instantaneous value, this is

$$\mathbf{H}(R,\theta;t) = \hat{\boldsymbol{\phi}}\frac{53}{R}\sin\theta\cos(6\pi\times10^{8}t - 2\pi R) \quad (\mu\text{A/m}).$$

Problem 6.26 A Hertzian dipole is a short conducting wire carrying an approximately constant current over its length l. If such a dipole is placed along the z-axis with its midpoint at the origin and if the current flowing through it is $i(t) = I_0\cos\omega t$, find

(a) the retarded vector potential $\widetilde{\mathbf{A}}(R,\theta,\phi)$ at an observation point $Q(R,\theta,\phi)$ in a spherical coordinate system, and

(b) the magnetic field phasor $\widetilde{\mathbf{H}}(R,\theta,\phi)$.

Assume l to be sufficiently small so that the observation point is approximately equidistant to all points on the dipole; that is, assume that $R' \simeq R$.

Solution:

(a) In phasor form, the current is given by $\widetilde{I} = I_0$. Explicitly writing the volume integral in Eq. (6.84) as a double integral over the wire cross section and a single integral over its length,

$$\widetilde{\mathbf{A}} = \frac{\mu}{4\pi}\int_{-l/2}^{l/2}\iint_s \frac{\widetilde{\mathbf{J}}(\mathbf{R}_\text{i})\,e^{-jkR'}}{R'}\,ds\,dz,$$

where s is the wire cross section. The wire is infinitesimally thin, so that R' is not a function of x or y and the integration over the cross section of the wire applies only to the current density. Recognizing that $\widetilde{J} = \hat{\mathbf{z}}I_0/s$, and employing the relation $R' \approx R$,

$$\widetilde{\mathbf{A}} = \hat{\mathbf{z}}\frac{\mu I_0}{4\pi}\int_{-l/2}^{l/2}\frac{e^{-jkR'}}{R'}\,dz \approx \hat{\mathbf{z}}\frac{\mu I_0}{4\pi}\int_{-l/2}^{l/2}\frac{e^{-jkR}}{R}\,dz = \hat{\mathbf{z}}\frac{\mu I_0 l}{4\pi R}e^{-jkR}.$$

In spherical coordinates, $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$, and therefore

$$\widetilde{\mathbf{A}} = (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\frac{\mu I_0 l}{4\pi R}e^{-jkR}.$$

(b) From Eq. (6.85),

$$\begin{aligned}
\widetilde{\mathbf{H}} = \frac{1}{\mu}\nabla\times\widetilde{\mathbf{A}} &= \frac{I_0 l}{4\pi}\nabla\times\left[(\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\frac{e^{-jkR}}{R}\right] \\
&= \frac{I_0 l}{4\pi}\hat{\boldsymbol{\phi}}\frac{1}{R}\left(\frac{\partial}{\partial R}\left(-\sin\theta e^{-jkR}\right) - \frac{\partial}{\partial\theta}\left(\cos\theta\frac{e^{-jkR}}{R}\right)\right) \\
&= \hat{\boldsymbol{\phi}}\frac{I_0 l\sin\theta e^{-jkR}}{4\pi R}\left(jk + \frac{1}{R}\right).
\end{aligned}$$

Problem 6.27 The magnetic field in a given dielectric medium is given by

$$\mathbf{H} = \hat{\mathbf{y}}\,6\cos 2z\sin(2\times 10^7 t - 0.1x) \quad \text{(A/m)},$$

where x and z are in meters. Determine:

(a) $\mathbf{E}$,

(b) the displacement current density $\mathbf{J}_\mathrm{d}$, and

(c) the charge density ρ_v.

Solution:

(a)

$$\begin{aligned}
\mathbf{H} &= \hat{\mathbf{y}}\,6\cos 2z\sin(2\times 10^7 t - 0.1x) = \hat{\mathbf{y}}\,6\cos 2z\cos(2\times 10^7 t - 0.1x - \pi/2), \\
\widetilde{\mathbf{H}} &= \hat{\mathbf{y}}\,6\cos 2z\,e^{-j0.1x}e^{-j\pi/2} = -\hat{\mathbf{y}}\,j6\cos 2z\,e^{-j0.1x}, \\
\widetilde{\mathbf{E}} &= \frac{1}{j\omega\varepsilon}\nabla\times\widetilde{\mathbf{H}} \\
&= \frac{1}{j\omega\varepsilon}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -j6\cos 2z\,e^{-j0.1x} & 0 \end{vmatrix} \\
&= \frac{1}{j\omega\varepsilon}\left\{\hat{\mathbf{x}}\left[-\frac{\partial}{\partial z}(-j6\cos 2z\,e^{-j0.1x})\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(-j6\cos 2z\,e^{-j0.1x})\right]\right\} \\
&= \hat{\mathbf{x}}\left(-\frac{12}{\omega\varepsilon}\sin 2z\,e^{-j0.1x}\right) + \hat{\mathbf{z}}\left(\frac{j0.6}{\omega\varepsilon}\cos 2z\,e^{-j0.1x}\right).
\end{aligned}$$

From the given expression for $\mathbf{H}$,

$$\omega = 2\times 10^7 \quad \text{(rad/s)},$$

$$\beta = 0.1 \quad \text{(rad/m)}.$$

Hence,

$$u_\text{p} = \frac{\omega}{\beta} = 2 \times 10^8 \quad \text{(m/s)},$$

and

$$\varepsilon_\text{r} = \left(\frac{c}{u_\text{p}}\right)^2 = \left(\frac{3\times 10^8}{2\times 10^8}\right)^2 = 2.25.$$

Using the values for ω and ε, we have

$$\widetilde{\mathbf{E}} = (-\hat{\mathbf{x}}30\sin 2z + \hat{\mathbf{z}}\,j1.5\cos 2z)\times 10^3 e^{-j0.1x} \quad \text{(V/m)},$$
$$\mathbf{E} = \left[-\hat{\mathbf{x}}30\sin 2z\cos(2\times 10^7 t - 0.1x) - \hat{\mathbf{z}}1.5\cos 2z\sin(2\times 10^7 t - 0.1x)\right] \quad \text{(kV/m)}.$$

(b)

$$\widetilde{\mathbf{D}} = \varepsilon\widetilde{\mathbf{E}} = \varepsilon_\text{r}\varepsilon_0\widetilde{\mathbf{E}} = (-\hat{\mathbf{x}}0.6\sin 2z + \hat{\mathbf{z}}\,j0.03\cos 2z)\times 10^{-6} e^{-j0.1x} \quad \text{(C/m}^2\text{)},$$
$$\mathbf{J}_\text{d} = \frac{\partial\mathbf{D}}{\partial t},$$

or

$$\widetilde{\mathbf{J}}_\text{d} = j\omega\widetilde{\mathbf{D}} = (-\hat{\mathbf{x}}\,j12\sin 2z - \hat{\mathbf{z}}0.6\cos 2z)e^{-j0.1x},$$
$$\begin{aligned}\mathbf{J}_\text{d} &= \mathfrak{Re}[\widetilde{\mathbf{J}}_\text{d}e^{j\omega t}] \\ &= \left[\hat{\mathbf{x}}12\sin 2z\sin(2\times 10^7 t - 0.1x) - \hat{\mathbf{z}}0.6\cos 2z\cos(2\times 10^7 t - 0.1x)\right] \quad \text{(A/m}^2\text{)}.\end{aligned}$$

(c) We can find ρ_v from

$$\nabla\cdot\mathbf{D} = \rho_\text{v}$$

or from

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_\text{v}}{\partial t}.$$

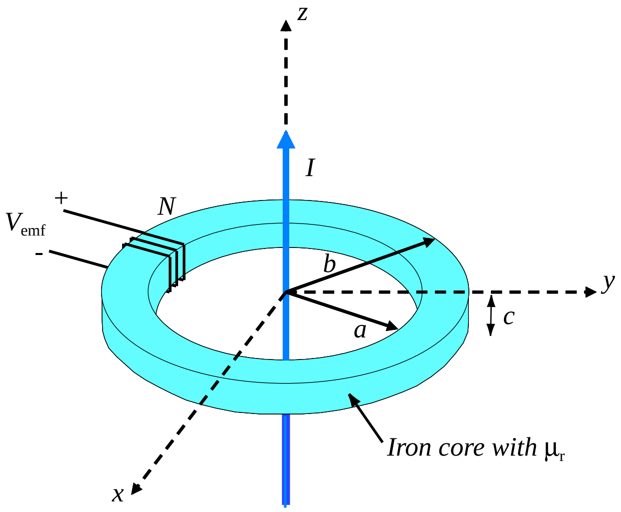
Applying Maxwell's equation,

$$\rho_\text{v} = \nabla\cdot\mathbf{D} = \varepsilon\nabla\cdot\mathbf{E} = \varepsilon_\text{r}\varepsilon_0\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z}\right)$$

yields

$$\begin{aligned}\rho_\text{v} = \varepsilon_\text{r}\varepsilon_0\Bigg\{&\frac{\partial}{\partial x}\left[-30\sin 2z\cos(2\times 10^7 t - 0.1x)\right] \\ &+ \frac{\partial}{\partial z}\left[-1.5\cos 2z\sin(2\times 10^7 t - 0.1x)\right]\Bigg\} \\ = \varepsilon_\text{r}\varepsilon_0&\left[-3\sin 2z\sin(2\times 10^7 t - 0.1x) + 3\sin 2z\sin(2\times 10^7 t - 0.1x)\right] = 0.\end{aligned}$$

Problem 6.28 The transformer shown in the figure consists of a long wire coincident with the z-axis carrying a current $I = I_0 \cos \omega t$, coupling magnetic energy to a toroidal coil situated in the x–y plane and centered at the origin. The toroidal core uses iron material with relative permeability μ_r, around which 100 turns of a tightly wound coil serves to induce a voltage V_{emf}, as shown in the figure.



(a) Develop an expression for V_{emf}.

(b) Calculate V_{emf} for $f = 60$ Hz, $\mu_r = 4000$, $a = 5$ cm, $b = 6$ cm, $c = 2$ cm, and $I_0 = 50$ A.

Solution:

(a) We start by calculating the magnetic flux through the coil, noting that r, the distance from the wire varies from a to b

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_a^b \hat{\mathbf{x}} \frac{\mu I}{2\pi r} \cdot \hat{\mathbf{x}} c \, dr = \frac{\mu c I}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$V_{\text{emf}} = -N \frac{d\Phi}{dt} = -\frac{\mu c N}{2\pi} \ln\left(\frac{b}{a}\right) \frac{dI}{dt}$$

$$= \frac{\mu c N \omega I_0}{2\pi} \ln\left(\frac{b}{a}\right) \sin \omega t \quad \text{(V)}.$$

(b)

$$V_{\text{emf}} = \frac{4000 \times 4\pi \times 10^{-7} \times 2 \times 10^{-2} \times 100 \times 2\pi \times 60 \times 50 \ln(6/5)}{2\pi} \sin 377t$$
$$= 5.5 \sin 377t \quad \text{(V)}.$$

Problem 6.29 In wet soil, characterized by $\sigma = 10^{-2}$ (S/m), $\mu_r = 1$, and $\varepsilon_r = 36$, at what frequency is the conduction current density equal in magnitude to the displacement current density?

Solution: For sinusoidal wave variation, the phasor electric field is

$$E = E_0 e^{j\omega t}.$$
$$J_c = \sigma E = \sigma E_0 e^{j\omega t}$$
$$J_d = \frac{\partial D}{\partial t} = \varepsilon \frac{\partial E}{\partial t} = j\omega\varepsilon E_0 e^{j\omega t}$$
$$\left|\frac{J_c}{J_d}\right| = 1 = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{2\pi\varepsilon f}$$

or

$$f = \frac{\sigma}{2\pi\varepsilon} = \frac{10^{-2}}{2\pi \times 36 \times 8.85 \times 10^{-12}} = 5 \times 10^6 = 5 \text{ MHz}.$$

Problem 6.30 In free space, the magnetic field is given by

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{36}{r} \cos(6 \times 10^9 t - kz) \quad \text{(mA/m)}.$$

(a) Determine k.

(b) Determine $\mathbf{E}$.

(c) Determine $\mathbf{J}_d$.

Solution:

(a) From the given expression, $\omega = 6 \times 10^9$ (rad/s), and since the medium is free space,

$$k = \frac{\omega}{c} = \frac{6 \times 10^9}{3 \times 10^8} = 20 \quad \text{(rad/m)}.$$

(b) Convert **H** to phasor:

$$\begin{aligned}
\widetilde{\mathbf{H}} &= \hat{\boldsymbol{\phi}}\,\frac{36}{r}\,e^{-jkz} \quad \text{(mA/m)}\\
\widetilde{\mathbf{E}} &= \frac{1}{j\omega\varepsilon_0}\,\nabla\times\widetilde{\mathbf{H}}\\
&= \frac{1}{j\omega\varepsilon_0}\left[-\hat{\mathbf{r}}\,\frac{\partial H_\phi}{\partial z}+\hat{\mathbf{z}}\,\frac{1}{r}\,\frac{\partial}{\partial r}(rH_\phi)\right]\\
&= \frac{1}{j\omega\varepsilon_0}\left[-\hat{\mathbf{r}}\,\frac{\partial}{\partial z}\left(\frac{36}{r}\,e^{-jkz}\right)+\frac{\hat{\mathbf{z}}}{r}\,\frac{\partial}{\partial r}(36e^{-jkz})\right]\\
&= \frac{1}{j\omega\varepsilon_0}\left[\hat{\mathbf{r}}\,\frac{j36k}{r}\,e^{-jkz}\right]\\
&= \hat{\mathbf{r}}\,\frac{36k}{\omega\varepsilon_0 r}\,e^{-jkz} = \hat{\mathbf{r}}\,\frac{36\times 377}{r}\,e^{-jkz}\times 10^{-3} = \hat{\mathbf{r}}\,\frac{13.6}{r}\,e^{-j20z} \quad \text{(V/m)}.\\
\mathbf{E} &= \mathfrak{Re}[\widetilde{E}e^{j\omega t}]\\
&= \hat{\mathbf{r}}\,\frac{13.6}{r}\cos(6\times 10^9 t - 20z) \quad \text{(V/m)}.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{J}_{\mathrm{d}} &= \varepsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\\
&= \hat{\mathbf{r}}\,\frac{13.6}{r}\,\varepsilon_0\,\frac{\partial}{\partial t}(\cos(6\times 10^9 t - 20z))\\
&= -\hat{\mathbf{r}}\,\frac{13.6\varepsilon_0\times 6\times 10^9}{r}\sin(6\times 10^9 t - 20z) \quad (\text{A/m}^2)\\
&= -\hat{\mathbf{r}}\,\frac{0.72}{r}\sin(6\times 10^9 t - 20z) \quad (\text{A/m}^2).
\end{aligned}$$

Chapter 7: Plane-Wave Propagation

Lesson #43

Chapter — Section: 7-1

Topics: Time-harmonic fields

Highlights:

- Phasors
- Complex permittivity
- Wave equations

Special Illustrations:

Lesson #44

Chapter — Section: 7-2

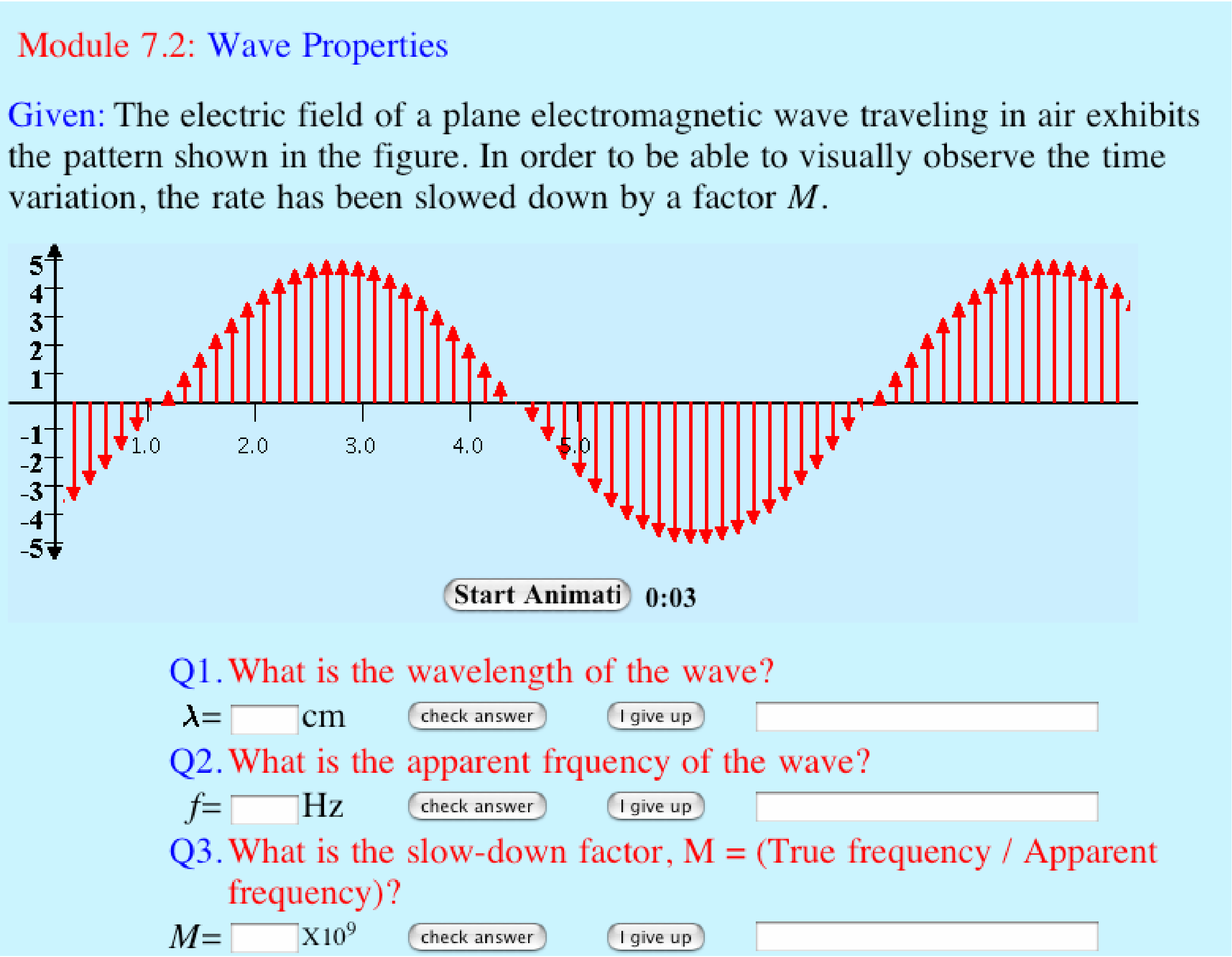
Topics: Waves in lossless media

Highlights:

- Uniform plane waves
- Intrinsic impedance
- Wave properties

Special Illustrations:

- Example 7-1
- CD-ROM Modules 7.3 and 7.4



Module 7.2: Wave Properties

Given: The electric field of a plane electromagnetic wave traveling in air exhibits the pattern shown in the figure. In order to be able to visually observe the time variation, the rate has been slowed down by a factor M.

Q1. What is the wavelength of the wave?

$\lambda =$ ☐ cm

Q2. What is the apparent frquency of the wave?

$f =$ ☐ Hz

Q3. What is the slow-down factor, M = (True frequency / Apparent frequency)?

$M =$ ☐ X10^9

Lesson #45 and 46

Chapter — Section: 7-3

Topics: Wave polarization

Highlights:

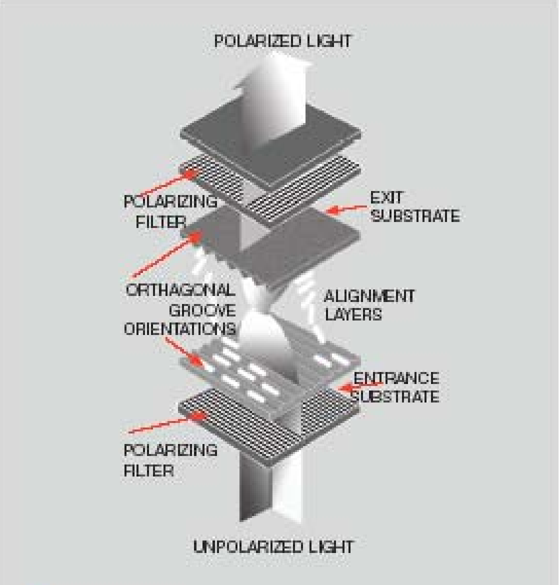
- Definition of polarization
- Linear, circular, elliptical

Special Illustrations:

- CD-ROM Demos 7.1-7.5
- Liquid Crystal Display

Liquid Crystal Display (LCD)

LCDs are used in digital clocks, cellular phones, desktop and laptop computers, and some televisions and other electronic systems. They offer a decided advantage over other display technologies, such as cathode ray tubes, in that they are much lighter and thinner and consume a lot less power to operate. LCD technology relies on special electrical and optical properties of a class of materials known as liquid crystals, first discovered in the 1880s by botanist Friedrich Reinitzer.



A. LCD structure

Physical Principle

Liquid crystals are neither a pure solid nor a pure liquid, but rather a hybrid of both. One particular variety of interest is the twisted nematic liquid crystal whose molecules have a natural tendency to assume a twisted spiral structure when the material is sandwiched between finely grooved glass substrates with orthogonal orientations (A). Note that the molecules in contact with the grooved surfaces align themselves in parallel along the grooves. The molecular spiral causes the crystal to behave like a wave polarizer; unpolarized light incident upon the entrance substrate follows the orientation of the spiral, emerging through the exit substrate with its polarization (direction of electric field) parallel to the groove's direction.

Lesson #47

Chapter — Section: 7-4

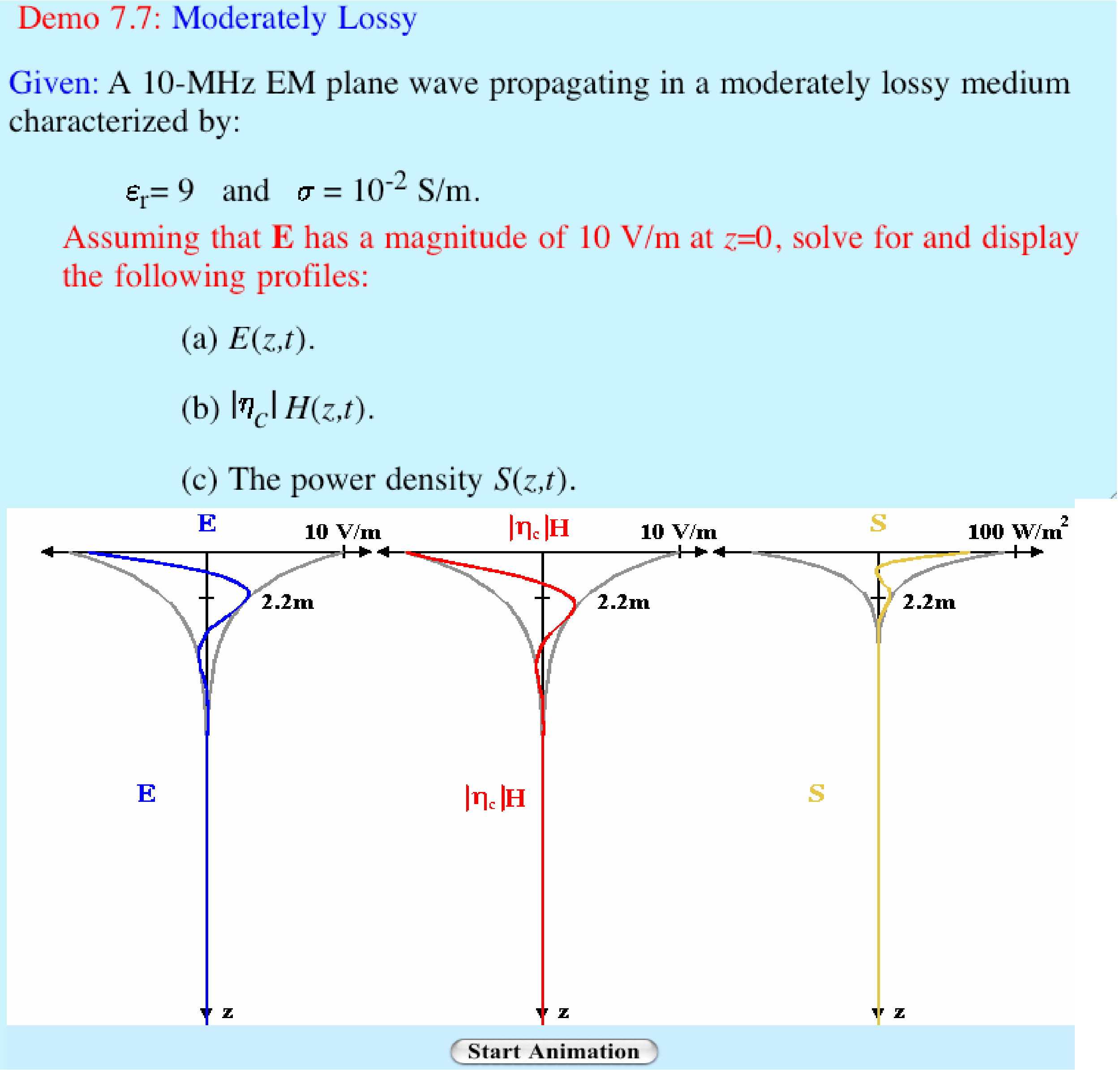
Topics: Waves in lossy media

Highlights:

- Attenuation and skin depth
- Low loss medium
- Good conductor

Special Illustrations:

- CD-ROM Demos 7.6-7.8



Demo 7.7: Moderately Lossy

Given: A 10-MHz EM plane wave propagating in a moderately lossy medium characterized by:

$\varepsilon_r = 9$ and $\sigma = 10^{-2}$ S/m.

Assuming that **E** has a magnitude of 10 V/m at z=0, solve for and display the following profiles:

(a) $E(z,t)$.

(b) $|\eta_c|\, H(z,t)$.

(c) The power density $S(z,t)$.

Lesson #48

Chapter — Section: 7-5

Topics: Current flow in conductors

Highlights:

- Skin depth dependence on frequency
- Surface impedance

Special Illustrations:

Lesson #49

Chapter — Section: 7-6

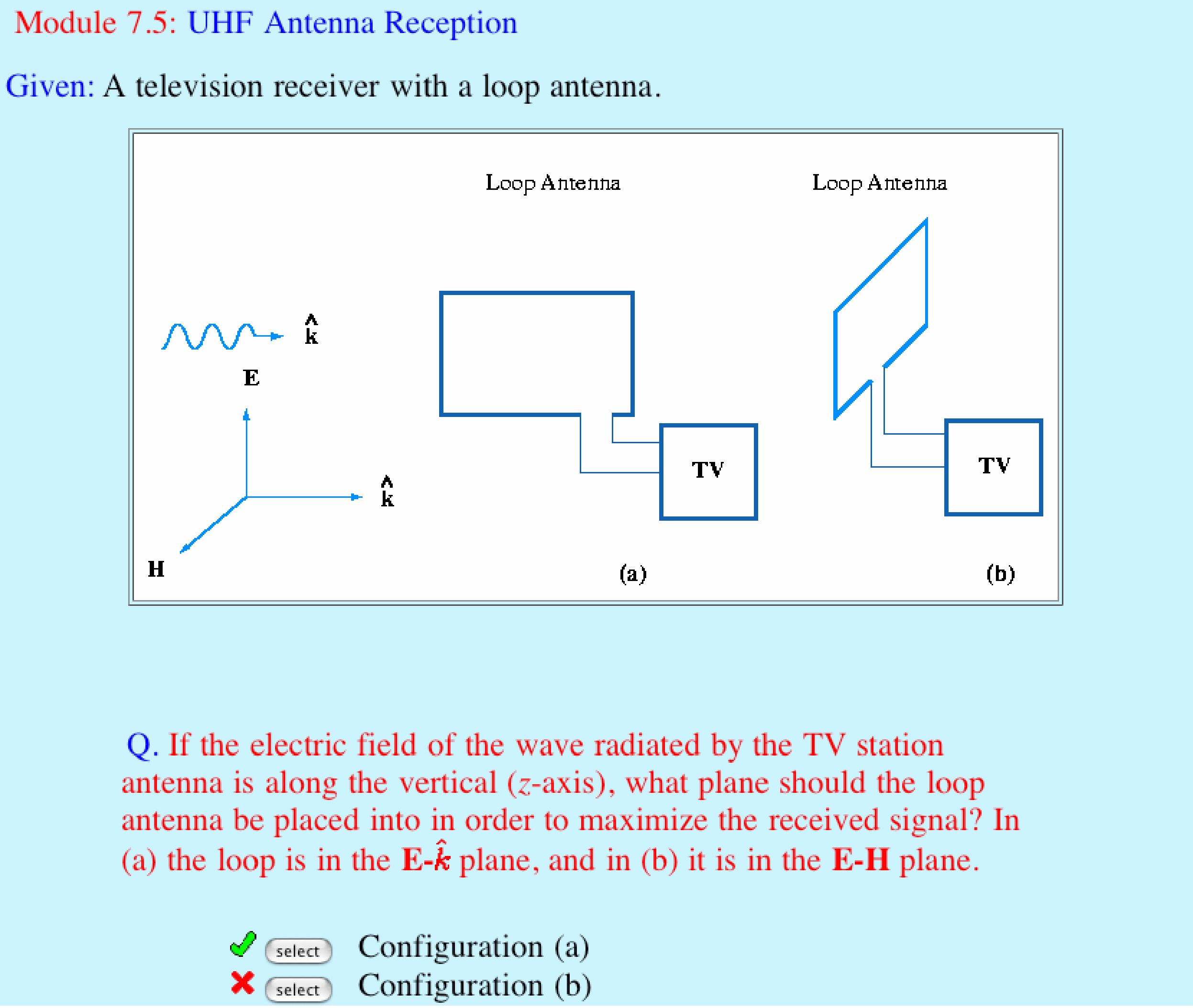
Topics: EM power density

Highlights:

- Power density in a lossless medium
- Power density in a lossy medium
- Time-average power

Special Illustrations:

- CD-ROM Module 7.5



Module 7.5: UHF Antenna Reception

Given: A television receiver with a loop antenna.

Q. If the electric field of the wave radiated by the TV station antenna is along the vertical (z-axis), what plane should the loop antenna be placed into in order to maximize the received signal? In (a) the loop is in the **E**-$\hat{k}$ plane, and in (b) it is in the **E**-**H** plane.

✓ Configuration (a)

✗ Configuration (b)

Chapter 7

Section 7-2: Propagation in Lossless Media

Problem 7.1 The magnetic field of a wave propagating through a certain nonmagnetic material is given by

$$\mathbf{H} = \hat{\mathbf{z}}\,30\cos(10^8 t - 0.5y) \quad \text{(mA/m)}.$$

Find (a) the direction of wave propagation, (b) the phase velocity, (c) the wavelength in the material, (d) the relative permittivity of the material, and (e) the electric field phasor.

Solution:

(a) Positive y-direction.

(b) $\omega = 10^8$ rad/s, $k = 0.5$ rad/m.

$$u_{\mathrm{p}} = \frac{\omega}{k} = \frac{10^8}{0.5} = 2 \times 10^8 \text{ m/s}.$$

(c) $\lambda = 2\pi/k = 2\pi/0.5 = 12.6$ m.

(d) $\varepsilon_{\mathrm{r}} = \left(\dfrac{c}{u_{\mathrm{p}}}\right)^2 = \left(\dfrac{3 \times 10^8}{2 \times 10^8}\right)^2 = 2.25.$

(e) From Eq. (7.39b),

$$\widetilde{\mathbf{E}} = -\eta\hat{\mathbf{k}} \times \widetilde{\mathbf{H}},$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{120\pi}{\sqrt{\varepsilon_{\mathrm{r}}}} = \frac{120\pi}{1.5} = 251.33 \quad (\Omega),$$

$$\hat{\mathbf{k}} = \hat{\mathbf{y}}, \quad \text{and} \quad \widetilde{\mathbf{H}} = \hat{\mathbf{z}}30e^{-j0.5y} \times 10^{-3} \quad \text{(A/m)}.$$

Hence,

$$\widetilde{\mathbf{E}} = -251.33\hat{\mathbf{y}} \times \hat{\mathbf{z}}30e^{-j0.5y} \times 10^{-3} = -\hat{\mathbf{x}}7.54e^{-j0.5y} \quad \text{(V/m)},$$

and

$$\mathbf{E}(y,t) = \mathfrak{Re}(\widetilde{\mathbf{E}}e^{j\omega t}) = -\hat{\mathbf{x}}7.54\cos(10^8 t - 0.5y) \quad \text{(V/m)}.$$

Problem 7.2 Write general expressions for the electric and magnetic fields of a 1-GHz sinusoidal plane wave traveling in the $+y$-direction in a lossless nonmagnetic medium with relative permittivity $\varepsilon_{\mathrm{r}} = 9$. The electric field is polarized along the x-direction, its peak value is 6 V/m and its intensity is 4 V/m at $t = 0$ and $y = 2$ cm.

Solution: For $f = 1$ GHz, $\mu_r = 1$, and $\varepsilon_r = 9$,

$$\omega = 2\pi f = 2\pi \times 10^9 \text{ rad/s},$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda_0}\sqrt{\varepsilon_r} = \frac{2\pi f}{c}\sqrt{\varepsilon_r} = \frac{2\pi \times 10^9}{3 \times 10^8}\sqrt{9} = 20\pi \text{ rad/m},$$

$$\mathbf{E}(y,t) = \hat{\mathbf{x}}6\cos(2\pi \times 10^9 t - 20\pi y + \phi_0) \quad \text{(V/m)}.$$

At $t = 0$ and $y = 2$ cm, $E = 4$ V/m:

$$4 = 6\cos(-20\pi \times 2 \times 10^{-2} + \phi_0) = 6\cos(-0.4\pi + \phi_0).$$

Hence,

$$\phi_0 - 0.4\pi = \cos^{-1}\left(\frac{4}{6}\right) = 0.84 \text{ rad},$$

which gives

$$\phi_0 = 2.1 \text{ rad} = 120.19^\circ$$

and

$$\mathbf{E}(y,t) = \hat{\mathbf{x}}\,6\cos(2\pi \times 10^9 t - 20\pi y + 120.19^\circ) \quad \text{(V/m)}.$$

Problem 7.3 The electric field phasor of a uniform plane wave is given by $\widetilde{\mathbf{E}} = \hat{\mathbf{y}}\,10e^{j0.2z}$ (V/m). If the phase velocity of the wave is 1.5×10^8 m/s and the relative permeability of the medium is $\mu_r = 2.4$, find (a) the wavelength, (b) the frequency f of the wave, (c) the relative permittivity of the medium, and (d) the magnetic field $\mathbf{H}(z,t)$.

Solution:

(a) From $\widetilde{\mathbf{E}} = \hat{\mathbf{y}}10e^{j0.2z}$ (V/m), we deduce that $k = 0.2$ rad/m. Hence,

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.2} = 10\pi = 31.42 \text{ m}.$$

(b)

$$f = \frac{u_p}{\lambda} = \frac{1.5 \times 10^8}{31.42} = 4.77 \times 10^6 \text{ Hz} = 4.77 \text{ MHz}.$$

(c) From

$$u_p = \frac{c}{\sqrt{\mu_r \varepsilon_r}}, \qquad \varepsilon_r = \frac{1}{\mu_r}\left(\frac{c}{u_p}\right)^2 = \frac{1}{2.4}\left(\frac{3}{1.5}\right)^2 = 1.67.$$

(d)

$$\begin{aligned}
\eta &= \sqrt{\frac{\mu}{\varepsilon}} \simeq 120\pi\sqrt{\frac{\mu_r}{\varepsilon_r}} = 120\pi\sqrt{\frac{2.4}{1.67}} = 451.94 \quad (\Omega), \\
\widetilde{\mathbf{H}} &= \frac{1}{\eta}(-\hat{\mathbf{z}}) \times \widetilde{\mathbf{E}} = \frac{1}{\eta}(-\hat{\mathbf{z}}) \times \hat{\mathbf{y}}10e^{j0.2z} = \hat{\mathbf{x}}22.13e^{j0.2z} \quad \text{(mA/m)}, \\
\mathbf{H}(z,t) &= \hat{\mathbf{x}}22.13\cos(\omega t + 0.2z) \quad \text{(mA/m)},
\end{aligned}$$

with $\omega = 2\pi f = 9.54\pi \times 10^6$ rad/s.

Problem 7.4 The electric field of a plane wave propagating in a nonmagnetic material is given by

$$\mathbf{E} = [\hat{\mathbf{y}}3\sin(\pi \times 10^7 t - 0.2\pi x) + \hat{\mathbf{z}}4\cos(\pi \times 10^7 t - 0.2\pi x)] \quad \text{(V/m)}.$$

Determine (a) the wavelength, (b) ε_r, and (c) $\mathbf{H}$.

Solution:

(a) Since $k = 0.2\pi$,

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.2\pi} = 10 \text{ m}.$$

(b)

$$u_p = \frac{\omega}{k} = \frac{\pi \times 10^7}{0.2\pi} = 5 \times 10^7 \text{ m/s}.$$

But

$$u_p = \frac{c}{\sqrt{\varepsilon_r}}.$$

Hence,

$$\varepsilon_r = \left(\frac{c}{u_p}\right)^2 = \left(\frac{3 \times 10^8}{5 \times 10^7}\right)^2 = 36.$$

(c)

$$\begin{aligned}
\mathbf{H} = \frac{1}{\eta}\hat{\mathbf{k}} \times \mathbf{E} &= \frac{1}{\eta}\hat{\mathbf{x}} \times \left[\hat{\mathbf{y}}3\sin(\pi \times 10^7 t - 0.2\pi x) + \hat{\mathbf{z}}4\cos(\pi \times 10^7 t - 0.2\pi x)\right] \\
&= \hat{\mathbf{z}}\frac{3}{\eta}\sin(\pi \times 10^7 t - 0.2\pi x) - \hat{\mathbf{y}}\frac{4}{\eta}\cos(\pi \times 10^7 t - 0.2\pi x) \quad \text{(A/m)},
\end{aligned}$$

with

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} \simeq \frac{120\pi}{6} = 20\pi = 62.83 \quad (\Omega).$$

Problem 7.5 A wave radiated by a source in air is incident upon a soil surface, whereupon a part of the wave is transmitted into the soil medium. If the wavelength of the wave is 60 cm in air and 20 cm in the soil medium, what is the soil's relative permittivity? Assume the soil to be a very low loss medium.

Solution: From $\lambda = \lambda_0/\sqrt{\varepsilon_r}$,

$$\varepsilon_r = \left(\frac{\lambda_0}{\lambda}\right)^2 = \left(\frac{60}{20}\right)^2 = 9.$$

Problem 7.6 The electric field of a plane wave propagating in a lossless, nonmagnetic, dielectric material with $\varepsilon_r = 2.56$ is given by

$$\mathbf{E} = \hat{\mathbf{y}}\,20\cos(6\pi \times 10^9 t - kz) \quad \text{(V/m)}.$$

Determine:

(a) f, u_p, λ, k, and η, and

(b) the magnetic field $\mathbf{H}$.

Solution:

(a)

$$\begin{aligned}
\omega &= 2\pi f = 6\pi \times 10^9 \text{ rad/s},\\
f &= 3 \times 10^9 \text{ Hz} = 3 \text{ GHz},\\
u_p &= \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{2.56}} = 1.875 \times 10^8 \text{ m/s},\\
\lambda &= \frac{u_p}{f} = \frac{1.875 \times 10^8}{6 \times 10^9} = 3.12 \text{ cm},\\
k &= \frac{2\pi}{\lambda} = \frac{2\pi}{3.12 \times 10^{-2}} = 201.4 \text{ rad/m},\\
\eta &= \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{377}{\sqrt{2.56}} = \frac{377}{1.6} = 235.62\ \Omega.
\end{aligned}$$

(b)

$$
\begin{aligned}
\mathbf{H} &= -\hat{\mathbf{x}}\frac{20}{\eta}\cos(6\pi\times10^9 t - kz)\\
&= -\hat{\mathbf{x}}\frac{20}{235.62}\cos(6\pi\times10^9 t - 201.4z)\\
&= -\hat{\mathbf{x}}8.49\times10^{-2}\cos(6\pi\times10^9 t - 201.4z)\quad \text{(A/m)}.
\end{aligned}
$$

Section 7-3: Wave Polarization

Problem 7.7 An RHC-polarized wave with a modulus of 2 (V/m) is traveling in free space in the negative z-direction. Write down the expression for the wave's electric field vector, given that the wavelength is 6 cm.

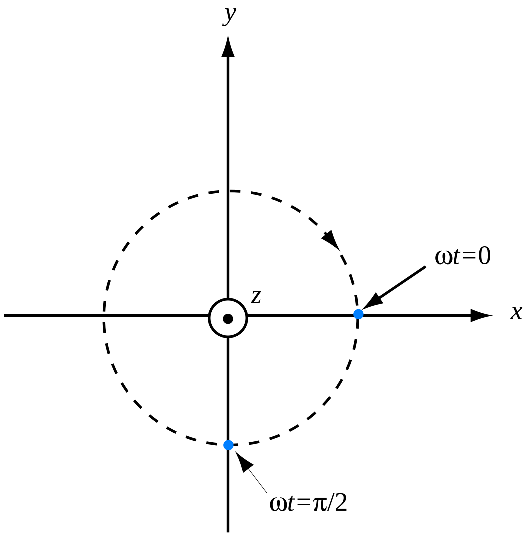


Figure P7.7: Locus of **E** versus time.

Solution: For an RHC wave traveling in $-\hat{\mathbf{z}}$, let us try the following:

$$\mathbf{E} = \hat{\mathbf{x}}a\cos(\omega t + kz) + \hat{\mathbf{y}}a\sin(\omega t + kz).$$

Modulus $|E| = \sqrt{a^2+a^2} = a\sqrt{2} = 2$ (V/m). Hence,

$$a = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Next, we need to check the sign of the $\hat{\mathbf{y}}$-component relative to that of the $\hat{\mathbf{x}}$-component. We do this by examining the locus of $\mathbf{E}$ versus t at $z = 0$: Since the wave is traveling along $-\hat{\mathbf{z}}$, when the thumb of the right hand is along $-\hat{\mathbf{z}}$ (into the page), the other four fingers point in the direction shown (clockwise as seen from above). Hence, we should reverse the sign of the $\hat{\mathbf{y}}$-component:

$$\mathbf{E} = \hat{\mathbf{x}}\sqrt{2}\cos(\omega t + kz) - \hat{\mathbf{y}}\sqrt{2}\sin(\omega t + kz) \quad \text{(V/m)}$$

with

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{6\times 10^{-2}} = 104.72 \quad \text{(rad/m)},$$

and

$$\omega = kc = \frac{2\pi}{\lambda}\times 3\times 10^{8} = \pi\times 10^{10} \quad \text{(rad/s)}.$$

Problem 7.8 For a wave characterized by the electric field

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}a_x\cos(\omega t - kz) + \hat{\mathbf{y}}a_y\cos(\omega t - kz + \delta),$$

identify the polarization state, determine the polarization angles (γ,χ), and sketch the locus of $\mathbf{E}(0,t)$ for each of the following cases:

(a) $a_x = 3$ V/m, $a_y = 4$ V/m, and $\delta = 0$,

(b) $a_x = 3$ V/m, $a_y = 4$ V/m, and $\delta = 180°$,

(c) $a_x = 3$ V/m, $a_y = 3$ V/m, and $\delta = 45°$,

(d) $a_x = 3$ V/m, $a_y = 4$ V/m, and $\delta = -135°$.

Solution:

$$\psi_0 = \tan^{-1}(a_y/a_x), \quad \text{[Eq. (7.60)]},$$

$$\tan 2\gamma = (\tan 2\psi_0)\cos\delta \quad \text{[Eq. (7.59a)]},$$

$$\sin 2\chi = (\sin 2\psi_0)\sin\delta \quad \text{[Eq. (7.59b)]}.$$

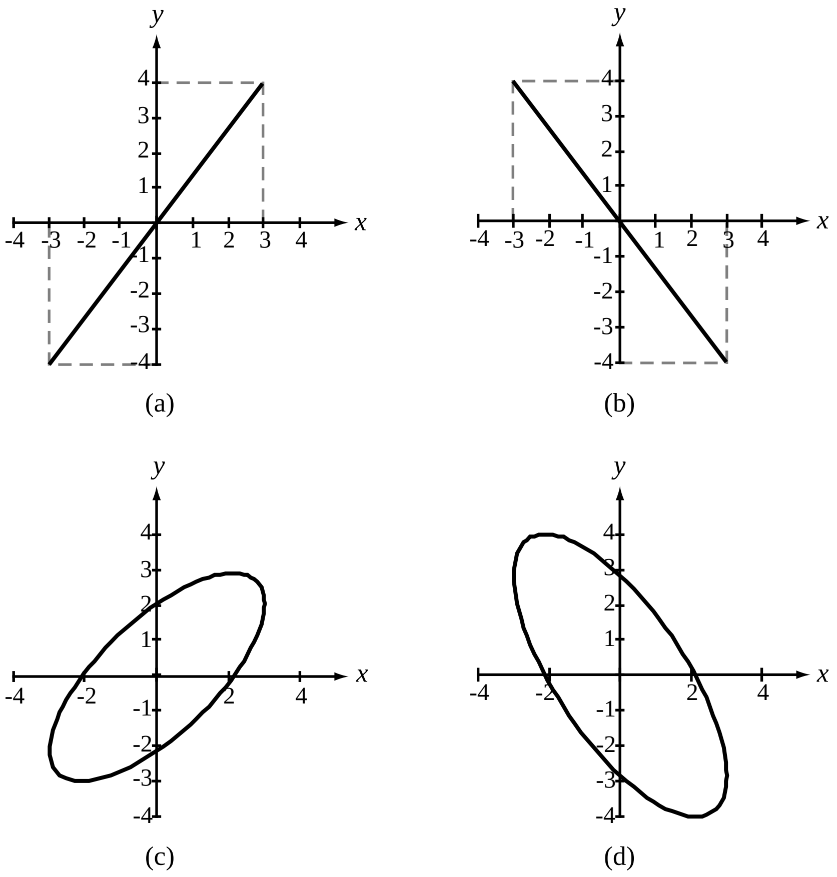
Case	a_x	a_y	δ	ψ_0	γ	χ	Polarization State
(a)	3	4	0	53.13°	53.13°	0	Linear
(b)	3	4	180°	53.13°	−53.13°	0	Linear
(c)	3	3	45°	45°	45°	22.5°	Left elliptical
(d)	3	4	−135°	53.13°	−56.2°	−21.37°	Right elliptical

(a) $\mathbf{E}(z,t) = \hat{\mathbf{x}}3\cos(\omega t - kz) + \hat{\mathbf{y}}4\cos(\omega t - kz)$.

(b) $\mathbf{E}(z,t) = \hat{\mathbf{x}}3\cos(\omega t - kz) - \hat{\mathbf{y}}4\cos(\omega t - kz)$.

(c) $\mathbf{E}(z,t) = \hat{\mathbf{x}}3\cos(\omega t - kz) + \hat{\mathbf{y}}3\cos(\omega t - kz + 45°)$.

(d) $\mathbf{E}(z,t) = \hat{\mathbf{x}}3\cos(\omega t - kz) + \hat{\mathbf{y}}4\cos(\omega t - kz - 135°)$.



(a) (b)

(c) (d)

Figure P7.8: Plots of the locus of $\mathbf{E}(0,t)$.

Problem 7.9 The electric field of a uniform plane wave propagating in free space is given by $\widetilde{\mathbf{E}} = (\hat{\mathbf{x}} + j\hat{\mathbf{y}})20e^{-j\pi z/6}$ (V/m). Specify the modulus and direction of the electric field intensity at the $z = 0$ plane at $t = 0$, 5 and 10 ns.

Solution:

$$\begin{aligned}
\mathbf{E}(z,t) &= \mathfrak{Re}[\widetilde{\mathbf{E}}e^{j\omega t}] \\
&= \mathfrak{Re}[(\hat{\mathbf{x}}+j\hat{\mathbf{y}})20e^{-j\pi z/6}e^{j\omega t}] \\
&= \mathfrak{Re}[(\hat{\mathbf{x}}+\hat{\mathbf{y}}e^{j\pi/2})20e^{-j\pi z/6}e^{j\omega t}] \\
&= \hat{\mathbf{x}}20\cos(\omega t-\pi z/6)+\hat{\mathbf{y}}20\cos(\omega t-\pi z/6+\pi/2) \\
&= \hat{\mathbf{x}}20\cos(\omega t-\pi z/6)-\hat{\mathbf{y}}20\sin(\omega t-\pi z/6) \quad \text{(V/m)}, \\
|\mathbf{E}| &= \left[E_x^2+E_y^2\right]^{1/2} = 20 \quad \text{(V/m)}, \\
\psi &= \tan^{-1}\left(\frac{E_y}{E_x}\right) = -(\omega t-\pi z/6).
\end{aligned}$$

From

$$\begin{aligned}
f &= \frac{c}{\lambda} = \frac{kc}{2\pi} = \frac{\pi/6\times 3\times 10^8}{2\pi} = 2.5\times 10^7 \text{ Hz}, \\
\omega &= 2\pi f = 5\pi\times 10^7 \text{ rad/s}.
\end{aligned}$$

At $z=0$,

$$\psi = -\omega t = -5\pi\times 10^7 t = \begin{cases} 0 & \text{at } t=0, \\ -0.25\pi = -45^\circ & \text{at } t = 5\,\text{ns}, \\ -0.5\pi = -90^\circ & \text{at } t = 10\,\text{ns}. \end{cases}$$

Therefore, the wave is LHC polarized.

Problem 7.10 A linearly polarized plane wave of the form $\widetilde{\mathbf{E}} = \hat{\mathbf{x}}a_x e^{-jkz}$ can be expressed as the sum of an RHC polarized wave with magnitude a_{R} and an LHC polarized wave with magnitude a_{L}. Prove this statement by finding expressions for a_{R} and a_{L} in terms of a_x.

Solution:

$$\begin{aligned}
& \widetilde{\mathbf{E}} = \hat{\mathbf{x}}a_x e^{-jkz}, \\
\text{RHC wave:}\quad & \widetilde{\mathbf{E}}_{\text{R}} = a_{\text{R}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}e^{-j\pi/2})e^{-jkz} = a_{\text{R}}(\hat{\mathbf{x}}-j\hat{\mathbf{y}})e^{-jkz}, \\
\text{LHC wave:}\quad & \widetilde{\mathbf{E}}_{\text{L}} = a_{\text{L}}(\hat{\mathbf{x}}+\hat{\mathbf{y}}e^{j\pi/2})e^{-jkz} = a_{\text{L}}(\hat{\mathbf{x}}+j\hat{\mathbf{y}})e^{-jkz}, \\
& \widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_{\text{R}}+\widetilde{\mathbf{E}}_{\text{L}}, \\
& \hat{\mathbf{x}}a_x = a_{\text{R}}(\hat{\mathbf{x}}-j\hat{\mathbf{y}})+a_{\text{L}}(\hat{\mathbf{x}}+j\hat{\mathbf{y}}).
\end{aligned}$$

By equating real and imaginary parts, $a_x = a_{\text{R}}+a_{\text{L}}$, $0 = -a_{\text{R}}+a_{\text{L}}$, or $a_{\text{L}} = a_x/2$, $a_{\text{R}} = a_x/2$.

Problem 7.11 The electric field of an elliptically polarized plane wave is given by

$$\mathbf{E}(z,t) = [-\hat{\mathbf{x}}\,10\sin(\omega t - kz - 60^\circ) + \hat{\mathbf{y}}\,30\cos(\omega t - kz)] \quad \text{(V/m)}.$$

Determine (a) the polarization angles (γ, χ) and (b) the direction of rotation.

Solution:

(a)

$$\begin{aligned}\mathbf{E}(z,t) &= [-\hat{\mathbf{x}}10\sin(\omega t - kz - 60^\circ) + \hat{\mathbf{y}}30\cos(\omega t - kz)] \\ &= [\hat{\mathbf{x}}10\cos(\omega t - kz + 30^\circ) + \hat{\mathbf{y}}30\cos(\omega t - kz)] \quad \text{(V/m)}.\end{aligned}$$

Phasor form:

$$\widetilde{\mathbf{E}} = (\hat{\mathbf{x}}10e^{j30^\circ} + \hat{\mathbf{y}}30)e^{-jkz}.$$

Since δ is defined as the phase of E_y relative to that of E_x,

$$\begin{aligned}\delta &= -30^\circ, \\ \psi_0 &= \tan^{-1}\left(\frac{30}{10}\right) = 71.56^\circ, \\ \tan 2\gamma &= (\tan 2\psi_0)\cos\delta = -0.65 \quad \text{or} \quad \gamma = 73.5^\circ, \\ \sin 2\chi &= (\sin 2\psi_0)\sin\delta = -0.40 \quad \text{or} \quad \chi = -8.73^\circ.\end{aligned}$$

(b) Since $\chi < 0$, the wave is right-hand elliptically polarized.

Problem 7.12 Compare the polarization states of each of the following pairs of plane waves:

(a) wave 1: $\mathbf{E}_1 = \hat{\mathbf{x}}2\cos(\omega t - kz) + \hat{\mathbf{y}}2\sin(\omega t - kz)$,
wave 2: $\mathbf{E}_2 = \hat{\mathbf{x}}2\cos(\omega t + kz) + \hat{\mathbf{y}}2\sin(\omega t + kz)$,

(b) wave 1: $\mathbf{E}_1 = \hat{\mathbf{x}}2\cos(\omega t - kz) - \hat{\mathbf{y}}2\sin(\omega t - kz)$,
wave 2: $\mathbf{E}_2 = \hat{\mathbf{x}}2\cos(\omega t + kz) - \hat{\mathbf{y}}2\sin(\omega t + kz)$.

Solution:

(a)

$$\begin{aligned}\mathbf{E}_1 &= \hat{\mathbf{x}}\,2\cos(\omega t - kz) + \hat{\mathbf{y}}\,2\sin(\omega t - kz) \\ &= \hat{\mathbf{x}}\,2\cos(\omega t - kz) + \hat{\mathbf{y}}\,2\cos(\omega t - kz - \pi/2), \\ \widetilde{\mathbf{E}}_1 &= \hat{\mathbf{x}}\,2e^{-jkz} + \hat{\mathbf{y}}\,2e^{-jkz}e^{-j\pi/2},\end{aligned}$$

$$\begin{aligned}\psi_0 &= \tan^{-1}\left(\frac{ay}{ax}\right) = \tan^{-1} 1 = 45^\circ,\\ \delta &= -\pi/2.\end{aligned}$$

Hence, wave 1 is RHC.

Similarly,

$$\widetilde{\mathbf{E}}_2 = \hat{\mathbf{x}}2e^{jkz} + \hat{\mathbf{y}}2e^{jkz}e^{-j\pi/2}.$$

Wave 2 has the same magnitude and phases as wave 1 except that its direction is along $-\hat{\mathbf{z}}$ instead of $+\hat{\mathbf{z}}$. Hence, the locus of rotation of **E** will match the left hand instead of the right hand. Thus, wave 2 is LHC.

(b)

$$\begin{aligned}\mathbf{E}_1 &= \hat{\mathbf{x}}2\cos(\omega t - kz) - \hat{\mathbf{y}}2\sin(\omega t - kz),\\ \widetilde{\mathbf{E}}_1 &= \hat{\mathbf{x}}2e^{-jkz} + \hat{\mathbf{y}}2e^{-jkz}e^{j\pi/2}.\end{aligned}$$

Wave 1 is LHC.

$$\widetilde{\mathbf{E}}_2 = \hat{\mathbf{x}}2e^{jkz} + \hat{\mathbf{y}}2e^{jkz}e^{j\pi/2}.$$

Reversal of direction of propagation (relative to wave 1) makes wave 2 RHC.

Problem 7.13 Plot the locus of $\mathbf{E}(0,t)$ for a plane wave with

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\sin(\omega t + kz) + \hat{\mathbf{y}}2\cos(\omega t + kz).$$

Determine the polarization state from your plot.

Solution:

$$\mathbf{E} = \hat{\mathbf{x}}\sin(\omega t + kz) + \hat{\mathbf{y}}2\cos(\omega t + kz).$$

Wave direction is $-\hat{\mathbf{z}}$. At $z = 0$,

$$\mathbf{E} = \hat{\mathbf{x}}\sin\omega t + \hat{\mathbf{y}}2\cos\omega t.$$

Tip of **E** rotates in accordance with right hand (with thumb pointing along $-\hat{\mathbf{z}}$). Hence, wave state is RHE.

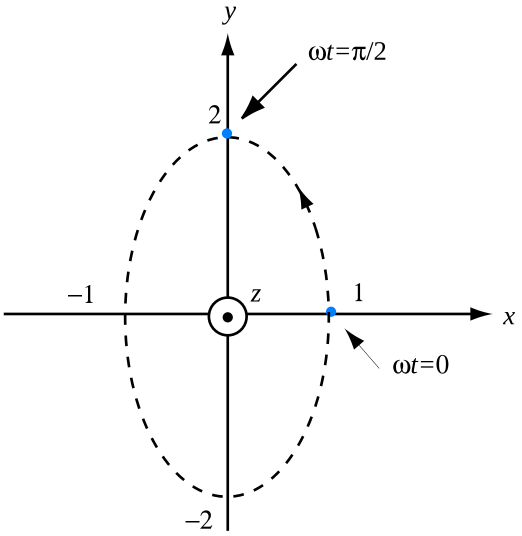


Figure P7.13: Locus of **E** versus time.

Sections 7-4: Propagation in a Lossy Medium

Problem 7.14 For each of the following combination of parameters, determine if the material is a low-loss dielectric, a quasi-conductor, or a good conductor, and then calculate α, β, λ, u_p, and η_c:

(a) glass with $\mu_r = 1$, $\varepsilon_r = 5$, and $\sigma = 10^{-12}$ S/m at 10 GHz,

(b) animal tissue with $\mu_r = 1$, $\varepsilon_r = 12$, and $\sigma = 0.3$ S/m at 100 MHz,

(c) wood with $\mu_r = 1$, $\varepsilon_r = 3$, and $\sigma = 10^{-4}$ S/m at 1 kHz.

Solution: Using equations given in Table 7-1:

	Case (a)	Case (b)	Case (c)
$\sigma/\omega\varepsilon$	3.6×10^{-13}	4.5	600
Type	low-loss dielectric	quasi-conductor	good conductor
α	8.42×10^{-11} Np/m	9.75 Np/m	6.3×10^{-4} Np/m
β	468.3 rad/m	12.16 rad/m	6.3×10^{-4} rad/m
λ	1.34 cm	51.69 cm	10 km
u_p	1.34×10^8 m/s	0.52×10^8 m/s	0.1×10^8 m/s
η_c	$\simeq 168.5\ \Omega$	$39.54 + j31.72\,\Omega$	$6.28(1 + j)\ \Omega$

Problem 7.15 Dry soil is characterized by $\varepsilon_r = 2.5$, $\mu_r = 1$, and $\sigma = 10^{-4}$ (S/m). At each of the following frequencies, determine if dry soil may be considered a good conductor, a quasi-conductor, or a low-loss dielectric, and then calculate α, β, λ, μ_p, and η_c:

(a) 60 Hz,
(b) 1 kHz,
(c) 1 MHz,
(d) 1 GHz.

Solution: $\varepsilon_r = 2.5$, $\mu_r = 1$, $\sigma = 10^{-4}$ S/m.

$f \rightarrow$	60 Hz	1 kHz	1 MHz	1 GHz
$\dfrac{\varepsilon''}{\varepsilon'} = \dfrac{\sigma}{\omega\varepsilon} = \dfrac{\sigma}{2\pi f \varepsilon_r \varepsilon_0}$	1.2×10^4	720	0.72	7.2×10^{-4}
Type of medium	Good conductor	Good conductor	Quasi-conductor	Low-loss dielectric
α (Np/m)	1.54×10^{-4}	6.28×10^{-4}	1.13×10^{-2}	1.19×10^{-2}
β (rad/m)	1.54×10^{-4}	6.28×10^{-4}	3.49×10^{-2}	33.14
λ (m)	4.08×10^4	10^4	180	0.19
u_p (m/s)	2.45×10^6	10^7	1.8×10^8	1.9×10^8
η_c (Ω)	$1.54(1+j)$	$6.28(1+j)$	$204.28 + j65.89$	238.27

Problem 7.16 In a medium characterized by $\varepsilon_r = 9$, $\mu_r = 1$, and $\sigma = 0.1$ S/m, determine the phase angle by which the magnetic field leads the electric field at 100 MHz.

Solution: The phase angle by which the magnetic field leads the electric field is $-\theta_\eta$ where θ_η is the phase angle of η_c.

$$\frac{\sigma}{\omega\varepsilon} = \frac{0.1 \times 36\pi}{2\pi \times 10^8 \times 10^{-9} \times 9} = 2.$$

Hence, quasi-conductor.

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon'}} \left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} = \frac{120\pi}{\sqrt{\varepsilon_r}} \left(1 - j\frac{\sigma}{\omega\varepsilon_0\varepsilon_r}\right)^{-1/2}$$
$$= 125.67(1 - j2)^{-1/2} = 71.49 + j44.18 = 84.04\angle 31.72^\circ.$$

Therefore $\theta_\eta = 31.72°$.

Since $\mathbf{H} = (1/\eta_c)\hat{\mathbf{k}} \times \mathbf{E}$, **H** leads **E** by $-\theta_\eta$, or by $-31.72°$. In other words, **H** lags **E** by $31.72°$.

Problem 7.17 Generate a plot for the skin depth δ_s versus frequency for seawater for the range from 1 kHz to 10 GHz (use log-log scales). The constitutive parameters of seawater are $\mu_r = 1$, $\varepsilon_r = 80$ and $\sigma = 4$ S/m.

Solution:

$$\delta_s = \frac{1}{\alpha} = \frac{1}{\omega}\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}-1\right]\right]^{-1/2},$$

$$\omega = 2\pi f,$$

$$\mu\varepsilon' = \mu_0\varepsilon_0\varepsilon_r = \frac{\varepsilon_r}{c^2} = \frac{80}{c^2} = \frac{80}{(3\times 10^8)^2},$$

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r} = \frac{4\times 36\pi}{2\pi f\times 10^{-9}\times 80} = \frac{72}{80f}\times 10^9.$$

See Fig. P7.17 for plot of δ_s versus frequency.

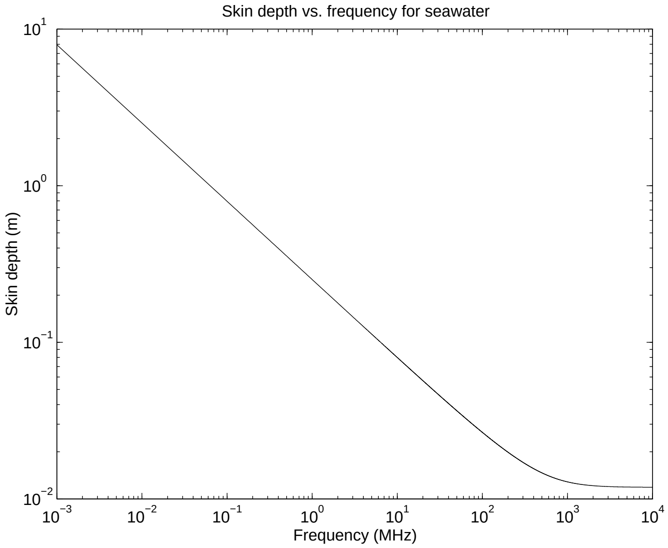


Figure P7.17: Skin depth versus frequency for seawater.

Problem 7.18 Ignoring reflection at the air-soil boundary, if the amplitude of a 3-GHz incident wave is 10 V/m at the surface of a wet soil medium, at what depth will it be down to 1 mV/m? Wet soil is characterized by $\mu_r = 1$, $\varepsilon_r = 9$, and $\sigma = 5 \times 10^{-4}$ S/m.

Solution:

$$E(z) = E_0 e^{-\alpha z} = 10 e^{-\alpha z},$$
$$\frac{\sigma}{\omega\varepsilon} = \frac{5 \times 10^{-4} \times 36\pi}{2\pi \times 3 \times 10^9 \times 10^{-9} \times 9} = 3.32 \times 10^{-4}.$$

Hence, medium is a low-loss dielectric.

$$\begin{aligned} \alpha &= \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma}{2} \cdot \frac{120\pi}{\sqrt{\varepsilon_r}} = \frac{5 \times 10^{-4} \times 120\pi}{2 \times \sqrt{9}} = 0.032 \quad \text{(Np/m)}, \\ 10^{-3} &= 10 e^{-0.032z}, \quad \ln 10^{-4} = -0.032z, \\ z &= 287.82 \text{ m.} \end{aligned}$$

Problem 7.19 The skin depth of a certain nonmagnetic conducting material is 3 μm at 5 GHz. Determine the phase velocity in the material.

Solution: For a good conductor, $\alpha = \beta$, and for any material $\delta_s = 1/\alpha$. Hence,

$$u_p = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = 2\pi f \delta_s = 2\pi \times 5 \times 10^9 \times 3 \times 10^{-6} = 9.42 \times 10^4 \quad \text{(m/s)}.$$

Problem 7.20 Based on wave attenuation and reflection measurements conducted at 1 MHz, it was determined that the intrinsic impedance of a certain medium is $28.1\angle 45^\circ$ (Ω) and the skin depth is 2 m. Determine (a) the conductivity of the material, (b) the wavelength in the medium, and (c) the phase velocity.

Solution:

(a) Since the phase angle of η_c is 45°, the material is a good conductor. Hence,

$$\eta_c = (1 + j)\frac{\alpha}{\sigma} = 28.1 e^{j45^\circ} = 28.1 \cos 45^\circ + j28.1 \sin 45^\circ,$$

or

$$\frac{\alpha}{\sigma} = 28.1 \cos 45^\circ = 19.87.$$

Since $\alpha = 1/\delta_s = 1/2 = 0.5$ Np/m,

$$\sigma = \frac{\alpha}{19.87} = \frac{0.5}{19.87} = 2.52 \times 10^{-2} \text{ S/m}.$$

(b) Since $\alpha = \beta$ for a good conductor, and $\alpha = 0.5$, it follows that $\beta = 0.5$. Therefore,

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.5} = 4\pi = 12.57 \text{ m}.$$

(c) $u_\text{p} = f\lambda = 10^6 \times 12.57 = 1.26 \times 10^7$ m/s.

Problem 7.21 The electric field of a plane wave propagating in a nonmagnetic medium is given by

$$\mathbf{E} = \hat{\mathbf{z}}\, 25e^{-30x} \cos(2\pi \times 10^9 t - 40x) \quad \text{(V/m)}.$$

Obtain the corresponding expression for **H**.

Solution: From the given expression for **E**,

$$\begin{aligned} \omega &= 2\pi \times 10^9 \quad \text{(rad/s)}, \\ \alpha &= 30 \quad \text{(Np/m)}, \\ \beta &= 40 \quad \text{(rad/m)}. \end{aligned}$$

From (7.65a) and (7.65b),

$$\begin{aligned} \alpha^2 - \beta^2 &= -\omega^2 \mu \varepsilon' = -\omega^2 \mu_0 \varepsilon_0 \varepsilon'_\text{r} = -\frac{\omega^2}{c^2}\, \varepsilon'_\text{r}, \\ 2\alpha\beta &= \omega^2 \mu \varepsilon'' = \frac{\omega^2}{c^2}\, \varepsilon''_\text{r}. \end{aligned}$$

Using the above values for ω, α, and β, we obtain the following:

$$\begin{aligned} \varepsilon'_\text{r} &= 1.6, \\ \varepsilon''_\text{r} &= 5.47. \end{aligned}$$

$$\begin{aligned} \eta_\text{c} &= \sqrt{\frac{\mu}{\varepsilon'}} \left(1 - j\,\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \\ &= \frac{\eta_0}{\sqrt{\varepsilon'_\text{r}}} \left(1 - j\,\frac{\varepsilon''_\text{r}}{\varepsilon'_\text{r}}\right)^{-1/2} = \frac{377}{\sqrt{1.6}} \left(1 - j\,\frac{5.47}{1.6}\right)^{-1/2} = 157.9\, e^{j36.85^\circ} \quad (\Omega). \end{aligned}$$

$$\widetilde{\mathbf{E}} = \hat{\mathbf{z}}25e^{-30x}e^{-j40x},$$
$$\widetilde{\mathbf{H}} = \frac{1}{\eta_c}\hat{\mathbf{k}}\times\widetilde{\mathbf{E}} = \frac{1}{157.9e^{j36.85^\circ}}\hat{\mathbf{x}}\times\hat{\mathbf{z}}25e^{-30x}e^{-j40x} = -\hat{\mathbf{y}}0.16e^{-30x}e^{-40x}e^{-j36.85^\circ},$$
$$\mathbf{H} = \mathfrak{Re}\{\widetilde{\mathbf{H}}e^{j\omega t}\} = -\hat{\mathbf{y}}0.16e^{-30x}\cos(2\pi\times 10^9 t - 40x - 36.85^\circ) \quad \text{(A/m)}.$$

Section 7-5: Current Flow in Conductors

Problem 7.22 In a nonmagnetic, lossy, dielectric medium, a 300-MHz plane wave is characterized by the magnetic field phasor

$$\widetilde{\mathbf{H}} = (\hat{\mathbf{x}} - j4\hat{\mathbf{z}})e^{-2y}e^{-j9y} \quad \text{(A/m)}.$$

Obtain time-domain expressions for the electric and magnetic field vectors.

Solution:

$$\widetilde{\mathbf{E}} = -\eta_c\hat{\mathbf{k}}\times\widetilde{\mathbf{H}}.$$

To find η_c, we need ε' and ε''. From the given expression for $\widetilde{\mathbf{H}}$,

$$\alpha = 2 \quad \text{(Np/m)},$$
$$\beta = 9 \quad \text{(rad/m)}.$$

Also, we are given than $f = 300$ MHz $= 3\times 10^8$ Hz. From (7.65a),

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon',$$
$$4 - 81 = -(2\pi\times 3\times 10^8)^2 \times 4\pi\times 10^{-7}\times\varepsilon'_r\times\frac{10^{-9}}{36\pi},$$

whose solution gives

$$\varepsilon'_r = 1.95.$$

Similarly, from (7.65b),

$$2\alpha\beta = \omega^2\mu\varepsilon'',$$
$$2\times 2\times 9 = (2\pi\times 3\times 10^8)^2\times 4\pi\times 10^{-7}\times\varepsilon''_r\times\frac{10^{-9}}{36\pi},$$

which gives

$$\varepsilon''_r = 0.91.$$

$$\begin{aligned}\eta_c &= \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\,\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}\\ &= \frac{\eta_0}{\sqrt{\varepsilon_r'}}\left(1 - j\,\frac{0.91}{1.95}\right)^{-1/2} = \frac{377}{\sqrt{1.95}}\,(0.93 + j0.21) = 256.9\,e^{j12.6^\circ}.\end{aligned}$$

Hence,

$$\begin{aligned}\widetilde{\mathbf{E}} &= -256.9\,e^{j12.6^\circ}\,\hat{\mathbf{y}} \times (\hat{\mathbf{x}} - j4\hat{\mathbf{z}})e^{-2y}e^{-j9y}\\ &= (\hat{\mathbf{x}}\,j4 + \hat{\mathbf{z}})\,256.9\,e^{-2y}e^{-j9y}e^{j12.6^\circ}\\ &= (\hat{\mathbf{x}}\,4e^{j\pi/2} + \hat{\mathbf{z}})\,256.9\,e^{-2y}e^{-j9y}e^{j12.6^\circ},\\ \mathbf{E} &= \mathfrak{Re}\{\widetilde{\mathbf{E}}e^{j\omega t}\}\\ &= \hat{\mathbf{x}}\,1.03\times10^{3}e^{-2y}\cos(\omega t - 9y + 102.6^\circ)\\ &\quad + \hat{\mathbf{z}}\,256.9\,e^{-2y}\cos(\omega t - 9y + 12.6^\circ) \quad \text{(V/m)},\\ \mathbf{H} &= \mathfrak{Re}\{\widetilde{\mathbf{H}}e^{j\omega t}\}\\ &= \mathfrak{Re}\{(\hat{\mathbf{x}} + j4\hat{\mathbf{z}})e^{-2y}e^{-j9y}e^{j\omega t}\}\\ &= \hat{\mathbf{x}}\,e^{-2y}\cos(\omega t - 9y) + \hat{\mathbf{z}}\,4e^{-2y}\sin(\omega t - 9y) \quad \text{(A/m)}.\end{aligned}$$

Problem 7.23 A rectangular copper block is 30 cm in height (along z). In response to a wave incident upon the block from above, a current is induced in the block in the positive x-direction. Determine the ratio of the a-c resistance of the block to its d-c resistance at 1 kHz. The relevant properties of copper are given in Appendix B.

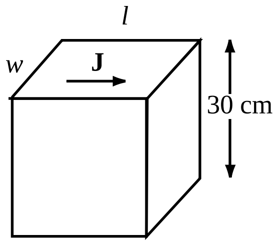


Figure P7.23: Copper block of Problem 7.23.

Solution:

$$\text{d-c resistance } R_{\text{dc}} = \frac{l}{\sigma A} = \frac{l}{0.3\sigma w},$$
$$\text{a-c resistance } R_{\text{ac}} = \frac{l}{\sigma w \delta_s}.$$

$$\frac{R_{\text{ac}}}{R_{\text{dc}}} = \frac{0.3}{\delta_s} = 0.3\sqrt{\pi f \mu \sigma} = 0.3[\pi \times 10^3 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7]^{1/2} = 143.55.$$

Problem 7.24 The inner and outer conductors of a coaxial cable have radii of 0.5 cm and 1 cm, respectively. The conductors are made of copper with $\varepsilon_r = 1$, $\mu_r = 1$ and $\sigma = 5.8 \times 10^7$ S/m, and the outer conductor is 0.5 mm thick. At 10 MHz:

- **(a)** Are the conductors thick enough to be considered infinitely thick so far as the flow of current through them is concerned?
- **(b)** Determine the surface resistance R_s.
- **(c)** Determine the a-c resistance per unit length of the cable.

Solution:

(a) From Eqs. (7.72) and (7.77b),

$$\delta_s = [\pi f \mu \sigma]^{-1/2} = [\pi \times 10^7 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7]^{-1/2} = 0.021 \text{ mm}.$$

Hence,

$$\frac{d}{\delta_s} = \frac{0.5 \text{ mm}}{0.021 \text{ mm}} \approx 25.$$

Hence, conductor is plenty thick.

(b) From Eq. (7.92a),

$$R_s = \frac{1}{\sigma \delta_s} = \frac{1}{5.8 \times 10^7 \times 2.1 \times 10^{-5}} = 8.2 \times 10^{-4}\ \Omega.$$

(c) From Eq. (7.96),

$$R' = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{8.2 \times 10^{-4}}{2\pi}\left(\frac{1}{5 \times 10^{-3}} + \frac{1}{10^{-2}}\right) = 0.039 \quad (\Omega/\text{m}).$$

Section 7-6: EM Power Density

Problem 7.25 The magnetic field of a plane wave traveling in air is given by $\mathbf{H} = \hat{\mathbf{x}}\,50\sin(2\pi \times 10^7 t - ky)$ (mA/m). Determine the average power density carried by the wave.

Solution:

$$\begin{aligned}
\mathbf{H} &= \hat{\mathbf{x}}50\sin(2\pi \times 10^7 t - ky) \quad \text{(mA/m)},\\
\mathbf{E} &= -\eta_0 \hat{\mathbf{y}} \times \mathbf{H} = \hat{\mathbf{z}}\eta_0 50\sin(2\pi \times 10^7 t - ky) \quad \text{(mV/m)},\\
\mathbf{S}_{\text{av}} &= (\hat{\mathbf{z}} \times \hat{\mathbf{x}})\frac{\eta_0 (50)^2}{2} \times 10^{-6} = \hat{\mathbf{y}}\frac{120\pi}{2}(50)^2 \times 10^{-6} = \hat{\mathbf{y}}0.48 \quad (\text{W/m}^2).
\end{aligned}$$

Problem 7.26 A wave traveling in a nonmagnetic medium with $\varepsilon_r = 9$ is characterized by an electric field given by

$$\mathbf{E} = [\hat{\mathbf{y}}\,3\cos(\pi \times 10^7 t + kx) - \hat{\mathbf{z}}\,2\cos(\pi \times 10^7 t + kx)] \quad \text{(V/m)}.$$

Determine the direction of wave travel and the average power density carried by the wave.

Solution:

$$\eta \simeq \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{120\pi}{\sqrt{9}} = 40\pi \quad (\Omega).$$

The wave is traveling in the negative x-direction.

$$\mathbf{S}_{\text{av}} = -\hat{\mathbf{x}}\frac{[3^2 + 2^2]}{2\eta} = -\hat{\mathbf{x}}\frac{13}{2 \times 40\pi} = -\hat{\mathbf{x}}0.05 \quad (\text{W/m}^2).$$

Problem 7.27 The electric-field phasor of a uniform plane wave traveling downward in water is given by

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\,5e^{-0.2z}e^{-j0.2z} \quad \text{(V/m)},$$

where $\hat{\mathbf{z}}$ is the downward direction and $z = 0$ is the water surface. If $\sigma = 4$ S/m,

- **(a)** obtain an expression for the average power density,
- **(b)** determine the attenuation rate, and
- **(c)** determine the depth at which the power density has been reduced by 40 dB.

Solution:

(a) Since $\alpha = \beta = 0.2$, the medium is a good conductor.

$$\eta_c = (1+j)\frac{\alpha}{\sigma} = (1+j)\frac{0.2}{4} = (1+j)0.05 = 0.0707e^{j45^\circ} \quad (\Omega).$$

From Eq. (7.109),

$$\mathbf{S}_{\text{av}} = \hat{\mathbf{z}}\frac{|E_0|^2}{2|\eta_c|}e^{-2\alpha z}\cos\theta_\eta = \hat{\mathbf{z}}\frac{25}{2\times 0.0707}e^{-0.4z}\cos 45^\circ = \hat{\mathbf{z}}125e^{-0.4z} \quad (\text{W/m}^2).$$

(b) $A = -8.68\alpha z = -8.68 \times 0.2z = -1.74z$ (dB).

(c) 40 dB is equivalent to 10^{-4}. Hence,

$$10^{-4} = e^{-2\alpha z} = e^{-0.4z}, \qquad \ln(10^{-4}) = -0.4z,$$

or $z = 23.03$ m.

Problem 7.28 The amplitudes of an elliptically polarized plane wave traveling in a lossless, nonmagnetic medium with $\varepsilon_r = 4$ are $H_{y0} = 3$ (mA/m) and $H_{z0} = 4$ (mA/m). Determine the average power flowing through an aperture in the y-z plane if its area is 20 m^2.

Solution:

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{120\pi}{\sqrt{4}} = 60\pi = 188.5\ \Omega,$$

$$\mathbf{S}_{\text{av}} = \hat{\mathbf{x}}\frac{\eta}{2}[H_{y0}^2 + H_{x0}^2] = \hat{\mathbf{x}}\frac{188.5}{2}[9+16]\times 10^{-6} = 2.36 \quad (\text{mW/m}^2),$$

$$P = S_{\text{av}}A = 2.36\times 10^{-3}\times 20 = 47.13 \quad (\text{mW}).$$

Problem 7.29 A wave traveling in a lossless, nonmagnetic medium has an electric field amplitude of 24.56 V/m and an average power density of 2.4 W/m^2. Determine the phase velocity of the wave.

Solution:

$$S_{\text{av}} = \frac{|E_0|^2}{2\eta}, \qquad \eta = \frac{|E_0|^2}{2S_{\text{av}}},$$

or

$$\eta = \frac{(24.56)^2}{2\times 2.4} = 125.67\ \Omega.$$

But

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{377}{\sqrt{\varepsilon_r}}, \qquad \varepsilon_r = \left(\frac{377}{125.67}\right)^2 = 9.$$

Hence,

$$u_p = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{3} = 1 \times 10^8 \text{ m/s}.$$

Problem 7.30 At microwave frequencies, the power density considered safe for human exposure is 1 (mW/cm^2). A radar radiates a wave with an electric field amplitude E that decays with distance as $E(R) = (3{,}000/R)$ (V/m), where R is the distance in meters. What is the radius of the unsafe region?

Solution:

$$S_{av} = \frac{|E(R)|^2}{2\eta_0}, \qquad 1 \text{ (mW/cm}^2) = 10^{-3} \text{ W/cm}^2 = 10 \text{ W/m}^2,$$

$$10 = \left(\frac{3 \times 10^3}{R}\right)^2 \times \frac{1}{2 \times 120\pi} = \frac{1.2 \times 10^4}{R^2},$$

$$R = \left(\frac{1.2 \times 10^4}{10}\right)^{1/2} = 34.64 \text{ m}.$$

Problem 7.31 Consider the imaginary rectangular box shown in Fig. 7-19 (P7.31).

(a) Determine the net power flux $P(t)$ entering the box due to a plane wave in air given by

$$\mathbf{E} = \hat{\mathbf{x}} E_0 \cos(\omega t - ky) \quad \text{(V/m)}.$$

(b) Determine the net time-average power entering the box.

Solution:

(a)

$$\mathbf{E} = \hat{\mathbf{x}} E_0 \cos(\omega t - ky),$$

$$\mathbf{H} = -\hat{\mathbf{z}} \frac{E_0}{\eta_0} \cos(\omega t - ky).$$

$$\mathbf{S}(t) = \mathbf{E} \times \mathbf{H} = \hat{\mathbf{y}} \frac{E_0^2}{\eta_0} \cos^2(\omega t - ky),$$

$$P(t) = S(t)\,A|_{y=0} - S(t)\,A|_{y=b} = \frac{E_0^2}{\eta_0} ac[\cos^2 \omega t - \cos^2(\omega t - kb)].$$

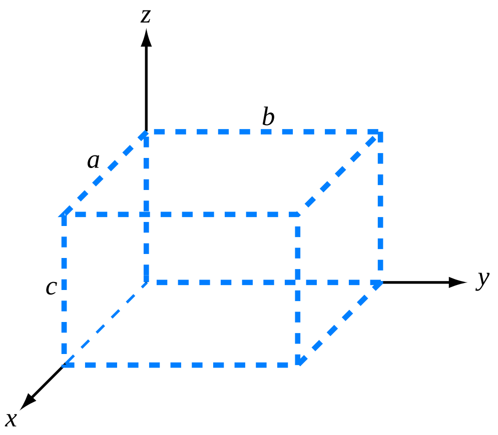


Figure P7.31: Imaginary rectangular box of Problems 7.31 and 7.32.

(b)

$$P_{\text{av}} = \frac{1}{T}\int_0^T P(t)\,dt.$$

where $T = 2\pi/\omega$.

$$P_{\text{av}} = \frac{E_0^2 ac}{\eta_0}\left\{\frac{\omega}{2\pi}\int_0^{2\pi/\omega}[\cos^2\omega t - \cos^2(\omega t - kb)]\,dt\right\} = 0.$$

Net average energy entering the box is zero, which is as expected since the box is in a lossless medium (air).

Problem 7.32 Repeat Problem 7.31 for a wave traveling in a lossy medium in which

$$\mathbf{E} = \hat{\mathbf{x}}100e^{-20y}\cos(2\pi\times 10^9 t - 40y) \quad \text{(V/m)},$$
$$\mathbf{H} = -\hat{\mathbf{z}}0.64e^{-20y}\cos(2\pi\times 10^9 t - 40y - 36.85^\circ) \quad \text{(A/m)}.$$

The box has dimensions $A = 1$ cm, $b = 2$ cm, and $c = 0.5$ cm.

Solution:

(a)

$$\begin{aligned}\mathbf{S}(t) &= \mathbf{E}\times\mathbf{H}\\ &= \hat{\mathbf{x}}100e^{-20y}\cos(2\pi\times 10^9 t - 40y)\\ &\quad\times(-\hat{\mathbf{z}}0.64)e^{-20y}\cos(2\pi\times 10^9 t - 40y - 36.85^\circ)\\ &= \hat{\mathbf{y}}64e^{-40y}\cos(2\pi\times 10^9 t - 40y)\cos(2\pi\times 10^9 t - 40y - 36.85^\circ).\end{aligned}$$

Using the identity $\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta+\phi)+\cos(\theta-\phi)]$,

$$\begin{aligned}
S(t) &= \frac{64}{2}e^{-40y}[\cos(4\pi\times 10^9 t - 80y - 36.85^\circ)+\cos 36.85^\circ],\\
P(t) &= S(t)\,A|_{y=0} - S(t)\,A|_{y=b}\\
&= 32ac\{[\cos(4\pi\times 10^9 t - 36.85^\circ)+\cos 36.85^\circ]\\
&\quad - e^{-40b}[\cos(4\pi\times 10^9 t - 80y - 36.85^\circ)+\cos 36.85^\circ]\}.
\end{aligned}$$

(b)

$$P_{\text{av}} = \frac{1}{T}\int_0^T P(t)\,dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} P(t)\,dt.$$

The average of $\cos(\omega t+\theta)$ over a period T is equal to zero, regardless of the value of θ. Hence,

$$P_{\text{av}} = 32ac(1-e^{-40b})\cos 36.85^\circ.$$

With $a = 1$ cm, $b = 2$ cm, and $c = 0.5$ cm,

$$P_{\text{av}} = 7.05\times 10^{-4} \quad \text{(W)}.$$

This is the average power absorbed by the lossy material in the box.

Problem 7.33 Given a wave with

$$\mathbf{E} = \hat{\mathbf{x}}E_0\cos(\omega t - kz),$$

calculate:

(a) the time-average electric energy density

$$(w_\text{e})_\text{av} = \frac{1}{T}\int_0^T w_\text{e}\,dt = \frac{1}{2T}\int_0^T \varepsilon E^2\,dt,$$

(b) the time-average magnetic energy density

$$(w_\text{m})_\text{av} = \frac{1}{T}\int_0^T w_\text{m}\,dt = \frac{1}{2T}\int_0^T \mu H^2\,dt,$$

and

(c) show that $(w_\text{e})_\text{av} = (w_\text{m})_\text{av}$.

Solution:

(a)

$$(w_\text{e})_\text{av} = \frac{1}{2T}\int_0^T \varepsilon E_0^2\cos^2(\omega t - kz)\,dt.$$

With $T = \frac{2\pi}{\omega}$,

$$
\begin{aligned}
(w_\mathrm{e})_\mathrm{av} &= \frac{\omega\varepsilon E_0^2}{4\pi}\int_0^{2\pi/\omega}\cos^2(\omega t - kz)\,dt \\
&= \frac{\varepsilon E_0^2}{4\pi}\int_0^{2\pi}\cos^2(\omega t - kz)\,d(\omega t) \\
&= \frac{\varepsilon E_0^2}{4}.
\end{aligned}
$$

(b)

$$\mathbf{H} = \hat{\mathbf{y}}\,\frac{E_0}{\eta}\cos(\omega t - kz).$$

$$
\begin{aligned}
(w_\mathrm{m})_\mathrm{av} &= \frac{1}{2T}\int_0^T \mu H^2\,dt \\
&= \frac{1}{2T}\int_0^T \mu\,\frac{E_0^2}{\eta^2}\cos^2(\omega t - kz)\,dt \\
&= \frac{\mu E_0^2}{4\eta^2}.
\end{aligned}
$$

(c)

$$(w_\mathrm{m})_\mathrm{av} = \frac{\mu E_0^2}{4\eta^2} = \frac{\mu E_0^2}{4\left(\frac{\mu}{\varepsilon}\right)} = \frac{\varepsilon E_0^2}{4} = (w_\mathrm{e})_\mathrm{av}.$$

Problem 7.34 A 60-MHz plane wave traveling in the $-x$-direction in dry soil with relative permittivity $\varepsilon_\mathrm{r} = 4$ has an electric field polarized along the z-direction. Assuming dry soil to be approximately lossless, and given that the magnetic field has a peak value of 10 (mA/m) and that its value was measured to be 7 (mA/m) at $t = 0$ and $x = -0.75$ m, develop complete expressions for the wave's electric and magnetic fields.

Solution: For $f = 60$ MHz $= 6\times10^7$ Hz, $\varepsilon_\mathrm{r} = 4$, $\mu_\mathrm{r} = 1$,

$$k = \frac{\omega}{c}\sqrt{\varepsilon_\mathrm{r}} = \frac{2\pi\times6\times10^7}{3\times10^8}\sqrt{4} = 0.8\pi \quad \text{(rad/m)}.$$

Given that **E** points along $\hat{\mathbf{z}}$ and wave travel is along $-\hat{\mathbf{x}}$, we can write

$$\mathbf{E}(x,t) = \hat{\mathbf{z}}E_0\cos(2\pi\times60\times10^6 t + 0.8\pi x + \phi_0) \quad \text{(V/m)}$$

where E_0 and ϕ_0 are unknown constants at this time. The intrinsic impedance of the medium is

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon_\mathrm{r}}} = \frac{120\pi}{2} = 60\pi \quad (\Omega).$$

With $\mathbf{E}$ along $\hat{\mathbf{z}}$ and $\hat{\mathbf{k}}$ along $-\hat{\mathbf{x}}$, (7.39) gives

$$\mathbf{H} = \frac{1}{\eta}\hat{\mathbf{k}} \times \mathbf{E}$$

or

$$\mathbf{H}(x,t) = \hat{\mathbf{y}}\frac{E_0}{\eta}\cos(1.2\pi \times 10^8 t + 0.8\pi x + \phi_0) \quad \text{(A/m)}.$$

Hence,

$$\frac{E_0}{\eta} = 10 \quad \text{(mA/m)}$$
$$E_0 = 10 \times 60\pi \times 10^{-3} = 0.6\pi \quad \text{(V/m)}.$$

Also,

$$H(-0.75\text{ m}, 0) = 7 \times 10^{-3} = 10\cos(-0.8\pi \times 0.75 + \phi_0) \times 10^{-3}$$

which leads to $\phi_0 = 153.6^\circ$.

Hence,

$$\mathbf{E}(x,t) = \hat{\mathbf{z}}\,0.6\pi\cos(1.2\pi \times 10^8 t + 0.8\pi x + 153.6^\circ) \quad \text{(V/m)}.$$
$$\mathbf{H}(x,t) = \hat{\mathbf{y}}\,10\cos(1.2\pi \times 10^8 t + 0.8\pi x + 153.6^\circ) \quad \text{(mA/m)}.$$

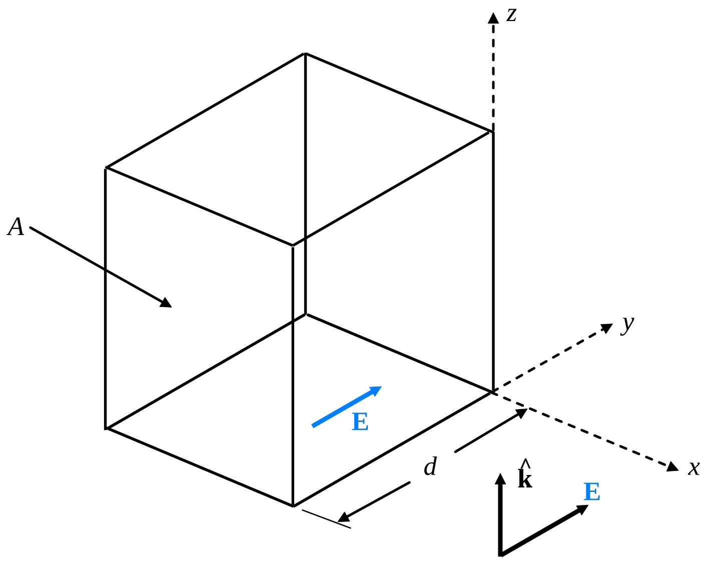
Problem 7.35 At 2 GHz, the conductivity of meat is on the order of 1 (S/m). When a material is placed inside a microwave oven and the field is activated, the presence of the electromagnetic fields in the conducting material causes energy dissipation in the material in the form of heat.

(a) Develop an expression for the time-average power per mm^3 dissipated in a material of conductivity σ if the peak electric field in the material is E_0.

(b) Evaluate the result for meat with $E_0 = 4 \times 10^4$ (V/m).

Solution:

(a) Let us consider a small volume of the material in the shape of a box of length d and cross sectional area A. Let us assume the microwave oven creates a wave traveling along the z direction with $\mathbf{E}$ along y, as shown.



Along y, the **E** field will create a voltage difference across the length of the box V, where

$$V = Ed.$$

Conduction current through the cross sectional area A is

$$I = JA = \sigma EA.$$

Hence, the instantaneous power is

$$\begin{aligned} P &= IV = \sigma E^2 (Ad) \\ &= \sigma E^2 \mathcal{V}. \end{aligned}$$

where $\mathcal{V} = Ad$ is the small volume under consideration. The power per mm^3 is obtained by setting $\mathcal{V} = (10^{-3})^3$,

$$P' = \frac{P}{10^{-9}} = \sigma E^2 \times 10^{-9} \quad (\text{W/mm}^3).$$

As a time harmonic signal, $E = E_0 \cos \omega t$. The time average dissipated power is

$$\begin{aligned} P'_{\text{av}} &= \left[\frac{1}{T} \int_0^T \sigma E_0^2 \cos^2 \omega t \, dt \right] \times 10^{-9} \\ &= \frac{1}{2} \sigma E_0^2 \times 10^{-9} \quad (\text{W/mm}^3). \end{aligned}$$

(b)

$$P'_{\text{av}} = \frac{1}{2} \times 1 \times (4 \times 10^4) 2 \times 10^{-9} = 0.8 \quad (\text{W/mm}^3).$$

Problem 7.36 A team of scientists is designing a radar as a probe for measuring the depth of the ice layer over the antarctic land mass. In order to measure a detectable echo due to the reflection by the ice-rock boundary, the thickness of the ice sheet should not exceed three skin depths. If $\varepsilon'_{\text{r}} = 3$ and $\varepsilon''_{\text{r}} = 10^{-2}$ for ice and if the maximum anticipated ice thickness in the area under exploration is 1.2 km, what frequency range is useable with the radar?

Solution:

$$3\delta_{\text{s}} = 1.2 \text{ km} = 1200 \text{ m}$$
$$\delta_{\text{s}} = 400 \text{ m}.$$

Hence,

$$\alpha = \frac{1}{\delta_{\text{s}}} = \frac{1}{400} = 2.5 \times 10^{-3} \quad (\text{Np/m}).$$

Since $\varepsilon''/\varepsilon' \ll 1$, we can use (7.75a) for α:

$$\alpha = \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} = \frac{2\pi f \varepsilon''_{\text{r}} \varepsilon_0}{2\sqrt{\varepsilon'_{\text{r}}}\sqrt{\varepsilon_0}}\sqrt{\mu_0} = \frac{\pi f \varepsilon''_{\text{r}}}{c\sqrt{\varepsilon_{\text{r}}}} = \frac{\pi f \times 10^{-2}}{3 \times 10^8 \sqrt{3}} = 6f \times 10^{-11} \text{Np/m}.$$

For $\alpha = 2.5 \times 10^{-3} = 6f \times 10^{-11}$,

$$f = 41.6 \text{ MHz}.$$

Since α increases with increasing frequency, the useable frequency range is

$$f \leq 41.6 \text{ MHz}.$$

Chapter 8: Reflection, Transmission, and Waveguides

Lessons #50 and 51

Chapter — Section: 8-1

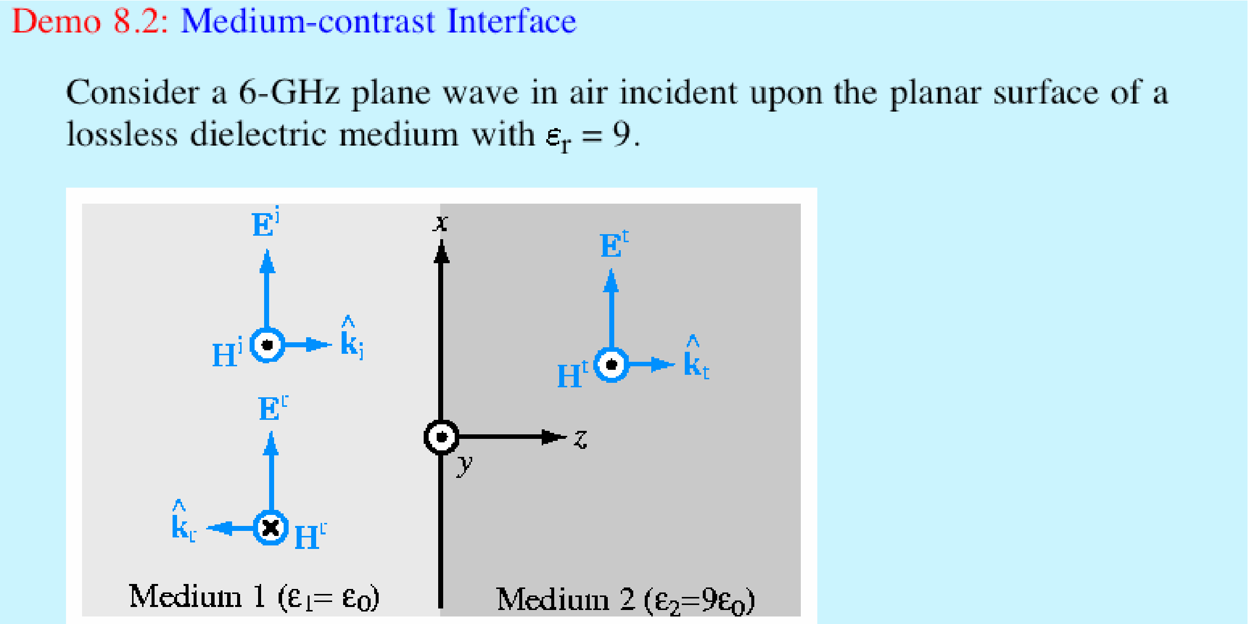
Topics: Normal incidence

Highlights:

- Analogy to transmission line
- Reflection and transmission coefficient

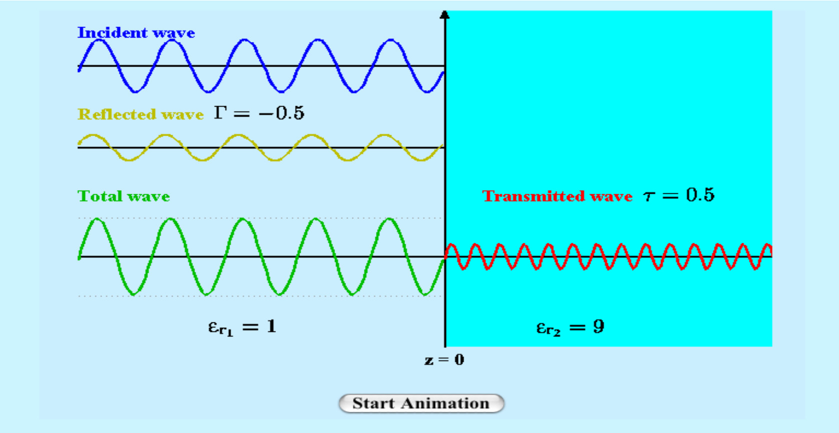
Special Illustrations:

- Example 8-1
- CD-ROM Modules 8.1-8.5
- CD-ROM Demos 8.2



Demo 8.2: Medium-contrast Interface

Consider a 6-GHz plane wave in air incident upon the planar surface of a lossless dielectric medium with $\varepsilon_r = 9$.



Lesson #52

Chapter — Section: 8-2

Topics: Snell's laws

Highlights:

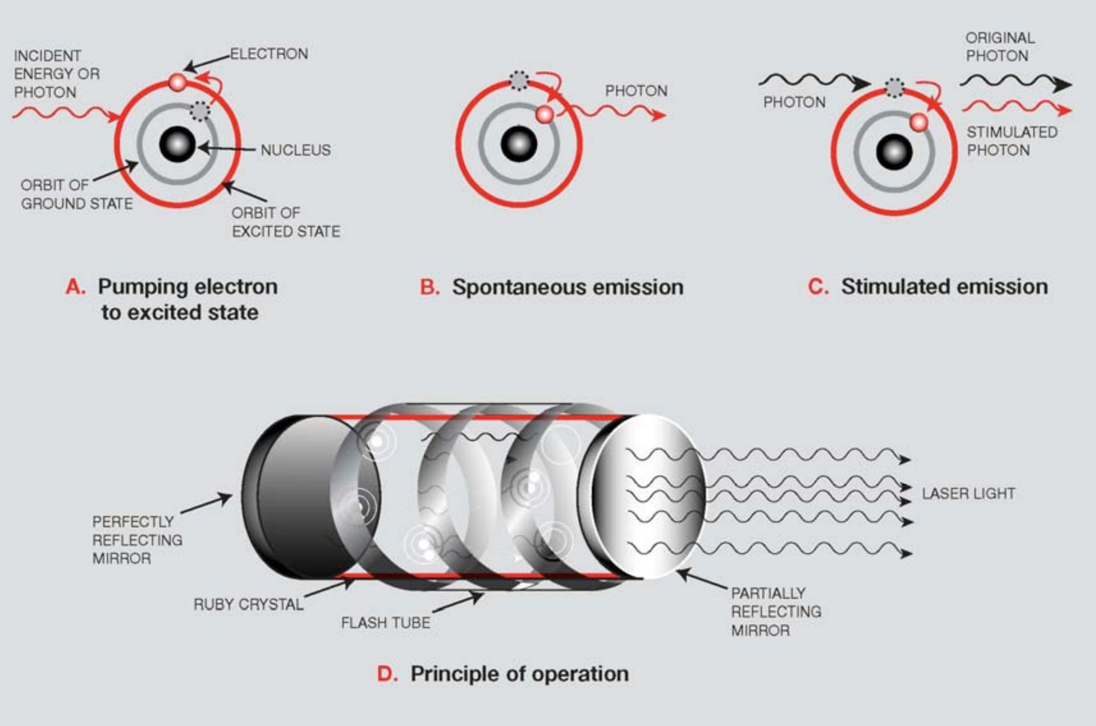
- Reflection and refraction
- Index of refraction

Special Illustrations:

- Example 8-4
- Technology Brief on "Lasers" (CD-ROM)

Lasers

Lasers are used in CD and DVD players, bar-code readers, eye surgery and multitudes of other systems and applications. A laser—acronym for light amplification by stimulated emission of **r**adiation—is a source of monochromatic (single wavelength), coherent (uniform wavefront), narrow-beam light, in contrast with other sources of light (such as the sun or a light bulb) which usually encompass waves of many different wavelengths with random phase (incoherent). A laser source generating microwaves is called a maser. The first maser was built in 1953 by Charles Townes and the first laser was constructed in 1960 by Theodore Maiman.



Lesson #53

Chapter — Section: 8-3

Topics: Fiber optics

Highlights:

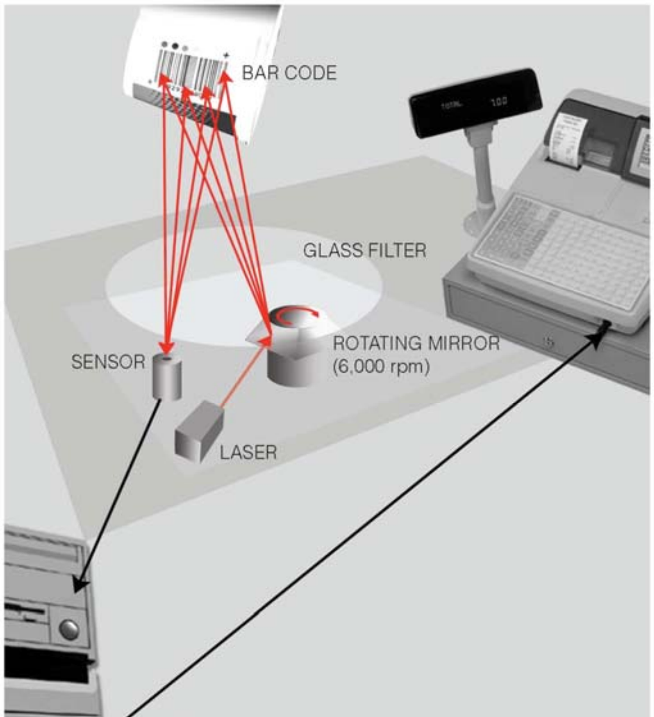
- Structure of an optical fiber
- Dispersion

Special Illustrations:

- Example 8-5
- Technology Brief on "Bar-Code Reader" (CD-ROM)

Bar Code Readers

A bar code consists of a sequence of parallel bars of certain widths, usually printed in black against a white background, configured to represent a particular binary code of information about a product and its manufacturer. Laser scanners can read the code and transfer the information to a computer, a cash register, or a display screen. For both stationary scanners built into checkout counters at grocery stores and handheld units that can be pointed at the bar-coded object like a gun, the basic operation of a bar-code reader is the same.



A. Elements of a bar code reader

Lessons #54 and 55

Chapter — Section: 8-4

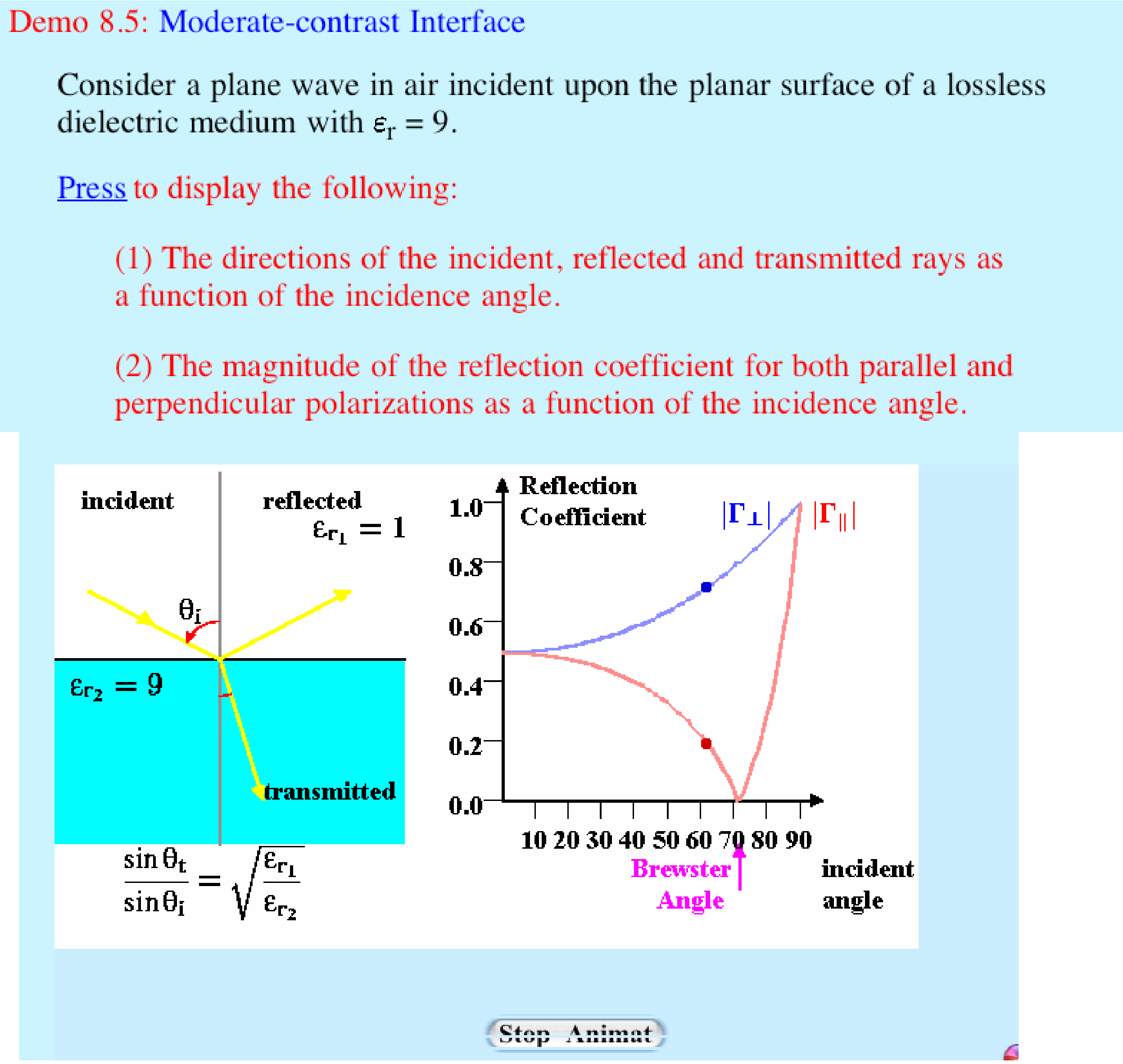
Topics: Oblique incidence

Highlights:

- Parallel and perpendicular polarizations
- Brewster angle
- Total internal reflection

Special Illustrations:

- Example 8-6 and 8-7
- CD-ROM Demos 8.4-8.6



Demo 8.5: Moderate-contrast Interface

Consider a plane wave in air incident upon the planar surface of a lossless dielectric medium with $\varepsilon_r = 9$.

Press to display the following:

(1) The directions of the incident, reflected and transmitted rays as a function of the incidence angle.

(2) The magnitude of the reflection coefficient for both parallel and perpendicular polarizations as a function of the incidence angle.

Lesson #56

Chapter — Section: 8-5

Topics: Reflectivity and transmissivity

Highlights:

- Power relations

Special Illustrations:

- Example 8-7

Lessons #57–59

Chapter — Section: 8-6 to 8-10

Topics: Waveguides

Highlights:

- TE and TM modes
- Cutoff frequency
- Phase and group velocities

Special Illustrations:

- Examples 8-8, 8-9, and 8-10

Lesson #60

Chapter — Section: 8-11

Topics: Cavity Resonators

Highlights:

- Resonant frequency
- Q factor
- Applications

Chapter 8

Section 8-1: Reflection and Transmission at Normal Incidence

Problem 8.1 A plane wave in air with an electric field amplitude of 20 V/m is incident normally upon the surface of a lossless, nonmagnetic medium with $\varepsilon_r = 25$. Determine:

(a) the reflection and transmission coefficients,

(b) the standing-wave ratio in the air medium, and

(c) the average power densities of the incident, reflected, and transmitted waves.

Solution:

(a)

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{120\pi}{5} = 24\pi \quad (\Omega).$$

From Eqs. (8.8a) and (8.9),

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{24\pi - 120\pi}{24\pi + 120\pi} = \frac{-96}{144} = -0.67,$$
$$\tau = 1 + \Gamma = 1 - 0.67 = 0.33.$$

(b)

$$S = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.67}{1-0.67} = 5.$$

(c) According to Eqs. (8.19) and (8.20),

$$S^{\rm i}_{\rm av} = \frac{|E^{\rm i}_0|^2}{2\eta_0} = \frac{400}{2\times 120\pi} = 0.52 \text{ W/m}^2,$$
$$S^{\rm r}_{\rm av} = |\Gamma|^2 S^{\rm i}_{\rm av} = (0.67)^2 \times 0.52 = 0.24 \text{ W/m}^2,$$
$$S^{\rm t}_{\rm av} = |\tau|^2 \frac{|E^{\rm i}_0|^2}{2\eta_2} = |\tau|^2 \frac{\eta_1}{\eta_2} S^{\rm i}_{\rm av} = (0.33)^2 \times \frac{120\pi}{24\pi} \times 0.52 = 0.28 \text{ W/m}^2.$$

Problem 8.2 A plane wave traveling in medium 1 with $\varepsilon_{r1} = 2.25$ is normally incident upon medium 2 with $\varepsilon_{r2} = 4$. Both media are made of nonmagnetic, non-conducting materials. If the electric field of the incident wave is given by

$$\mathbf{E}^{\rm i} = \hat{\mathbf{y}} 8 \cos(6\pi \times 10^9 t - 30\pi x) \quad \text{(V/m)},$$

(a) obtain time-domain expressions for the electric and magnetic fields in each of the two media, and

(b) determine the average power densities of the incident, reflected and transmitted waves.

Solution:

(a)

$$\begin{aligned}
\mathbf{E}^{\mathrm{i}} &= \hat{\mathbf{y}}\,8\cos(6\pi\times 10^9 t - 30\pi x) \quad \text{(V/m)},\\
\eta_1 &= \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_1}}} = \frac{\eta_0}{\sqrt{2.25}} = \frac{\eta_0}{1.5} = \frac{377}{1.5} = 251.33\ \Omega,\\
\eta_2 &= \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_2}}} = \frac{\eta_0}{\sqrt{4}} = \frac{377}{2} = 188.5\ \Omega,\\
\Gamma &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1/2 - 1/1.5}{1/2 + 1/1.5} = -0.143,\\
\tau &= 1 + \Gamma = 1 - 0.143 = 0.857,\\
\mathbf{E}^{\mathrm{r}} &= \Gamma\mathbf{E}^{\mathrm{i}} = -1.14\,\hat{\mathbf{y}}\cos(6\pi\times 10^9 t + 30\pi x) \quad \text{(V/m)}.
\end{aligned}$$

Note that the coefficient of x is positive, denoting the fact that $\mathbf{E}^{\mathrm{r}}$ belongs to a wave traveling in $-x$-direction.

$$\begin{aligned}
\mathbf{E}_1 &= \mathbf{E}^{\mathrm{i}} + \mathbf{E}^{\mathrm{r}} = \hat{\mathbf{y}}\,[8\cos(6\pi\times 10^9 t - 30\pi x) - 1.14\cos(6\pi\times 10^9 t + 30\pi x)] \quad \text{(A/m)},\\
\mathbf{H}^{\mathrm{i}} &= \hat{\mathbf{z}}\frac{8}{\eta_1}\cos(6\pi\times 10^9 t - 30\pi x) = \hat{\mathbf{z}}\,31.83\cos(6\pi\times 10^9 t - 30\pi x) \quad \text{(mA/m)},\\
\mathbf{H}^{\mathrm{r}} &= \hat{\mathbf{z}}\frac{1.14}{\eta_1}\cos(6\pi\times 10^9 t + 30\pi x) = \hat{\mathbf{z}}\,4.54\cos(6\pi\times 10^9 t + 30\pi x) \quad \text{(mA/m)},\\
\mathbf{H}_1 &= \mathbf{H}^{\mathrm{i}} + \mathbf{H}^{\mathrm{r}}\\
&= \hat{\mathbf{z}}\,[31.83\cos(6\pi\times 10^9 t - 30\pi x) + 4.54\cos(6\pi\times 10^9 t + 30\pi x)] \quad \text{(mA/m)}.
\end{aligned}$$

Since $k_1 = \omega\sqrt{\mu\varepsilon_1}$ and $k_2 = \omega\sqrt{\mu\varepsilon_2}$,

$$\begin{aligned}
k_2 &= \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\,k_1 = \sqrt{\frac{4}{2.25}}\,30\pi = 40\pi \quad \text{(rad/m)},\\
\mathbf{E}_2 &= \mathbf{E}^{\mathrm{t}} = \hat{\mathbf{y}}\,8\tau\cos(6\pi\times 10^9 t - 40\pi x) = \hat{\mathbf{y}}\,6.86\cos(6\pi\times 10^9 t - 40\pi x) \quad \text{(V/m)},\\
\mathbf{H}_2 &= \mathbf{H}^{\mathrm{t}} = \hat{\mathbf{z}}\,\frac{8\tau}{\eta_2}\cos(6\pi\times 10^9 t - 40\pi x) = \hat{\mathbf{z}}\,36.38\cos(6\pi\times 10^9 t - 40\pi x) \quad \text{(mA/m)}.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{S}^{\mathrm{i}}_{\mathrm{av}} &= \hat{\mathbf{x}}\,\frac{8^2}{2\eta_1} = \frac{64}{2\times 251.33} = \hat{\mathbf{x}}\,127.3 \quad (\text{mW/m}^2),\\
\mathbf{S}^{\mathrm{r}}_{\mathrm{av}} &= -|\Gamma|^2\mathbf{S}^{\mathrm{i}}_{\mathrm{av}} = -\hat{\mathbf{x}}\,(0.143)^2\times 0.127 = -\hat{\mathbf{x}}\,2.6 \quad (\text{mW/m}^2),
\end{aligned}$$

$$\begin{aligned}\mathbf{S}^{\mathrm{t}}_{\mathrm{av}} &= \frac{|E^{\mathrm{t}}_0|^2}{2\eta_2}\\ &= \hat{\mathbf{x}}\,\tau^2\frac{(8)^2}{2\eta_2} = \hat{\mathbf{x}}\,\frac{(0.86)^2 64}{2\times 188.5} = \hat{\mathbf{x}}\,124.7 \quad (\mathrm{mW/m^2}).\end{aligned}$$

Within calculation error, $\mathbf{S}^{\mathrm{i}}_{\mathrm{av}} + \mathbf{S}^{\mathrm{r}}_{\mathrm{av}} = \mathbf{S}^{\mathrm{t}}_{\mathrm{av}}$.

Problem 8.3 A plane wave traveling in a medium with $\varepsilon_{\mathrm{r}_1} = 9$ is normally incident upon a second medium with $\varepsilon_{\mathrm{r}_2} = 4$. Both media are made of nonmagnetic, non-conducting materials. If the magnetic field of the incident plane wave is given by

$$\mathbf{H}^{\mathrm{i}} = \hat{\mathbf{z}}\,2\cos(2\pi\times 10^9 t - ky) \quad (\mathrm{A/m}),$$

(a) obtain time domain expressions for the electric and magnetic fields in each of the two media, and

(b) determine the average power densities of the incident, reflected and transmitted waves.

Solution:

(a) In medium 1,

$$\begin{aligned}u_{\mathrm{p}} &= \frac{c}{\sqrt{\varepsilon_{\mathrm{r}_1}}} = \frac{3\times 10^8}{\sqrt{9}} = 1\times 10^8 \quad (\mathrm{m/s}),\\ k_1 &= \frac{\omega}{u_{\mathrm{p}}} = \frac{2\pi\times 10^9}{1\times 10^8} = 20\pi \quad (\mathrm{rad/m}),\\ \mathbf{H}^{\mathrm{i}} &= \hat{\mathbf{z}}\,2\cos(2\pi\times 10^9 t - 20\pi y) \quad (\mathrm{A/m}),\\ \eta_1 &= \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_1}}} = \frac{377}{3} = 125.67\ \Omega,\\ \eta_2 &= \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_2}}} = \frac{377}{2} = 188.5\ \Omega,\\ \mathbf{E}^{\mathrm{i}} &= -\hat{\mathbf{x}}\,2\eta_1\cos(2\pi\times 10^9 t - 20\pi y)\\ &= -\hat{\mathbf{x}}\,251.34\cos(2\pi\times 10^9 t - 20\pi y) \quad (\mathrm{V/m}),\end{aligned}$$

$$\begin{aligned}\Gamma &= \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{188.5-125.67}{188.5+125.67} = 0.2,\\ \tau &= 1+\Gamma = 1.2,\\ \mathbf{E}^{\mathrm{r}} &= -\hat{\mathbf{x}}\,251.34\times 0.2\cos(2\pi\times 10^9 t + 20\pi y)\\ &= -\hat{\mathbf{x}}\,50.27\cos(2\pi\times 10^9 t + 20\pi y) \quad (\mathrm{V/m}),\end{aligned}$$

$$\begin{aligned}\mathbf{H}^{\mathrm{r}} &= -\hat{\mathbf{z}}\,\frac{50.27}{\eta_1}\cos(2\pi\times10^9t+20\pi y)\\ &= -\hat{\mathbf{z}}\,0.4\cos(2\pi\times10^9t+20\pi y)\quad(\mathrm{A/m}),\end{aligned}$$

$$\begin{aligned}\mathbf{E}_1 &= \mathbf{E}^{\mathrm{i}}+\mathbf{E}^{\mathrm{r}}\\ &= -\hat{\mathbf{x}}\,[25.134\cos(2\pi\times10^9t-20\pi y)+50.27\cos(2\pi\times10^9t+20\pi y)]\quad(\mathrm{V/m}),\\ \mathbf{H}_1 &= \mathbf{H}^{\mathrm{i}}+\mathbf{H}^{\mathrm{r}} = \hat{\mathbf{z}}\,[2\cos(2\pi\times10^9t-20\pi y)-0.4\cos(2\pi\times10^9t+20\pi y)]\quad(\mathrm{A/m}).\end{aligned}$$

In medium 2,

$$\begin{aligned}k_2 &= \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\,k_1 = \sqrt{\frac{4}{9}}\times20\pi = \frac{40\pi}{3}\quad(\mathrm{rad/m}),\\ \mathbf{E}_2 = \mathbf{E}^{\mathrm{t}} &= -\hat{\mathbf{x}}\,251.34\,\tau\cos\left(2\pi\times10^9t-\frac{40\pi y}{3}\right)\\ &= -\hat{\mathbf{x}}\,301.61\cos\left(2\pi\times10^9t-\frac{40\pi y}{3}\right)\quad(\mathrm{V/m}),\\ \mathbf{H}_2 = \mathbf{H}^{\mathrm{t}} &= \hat{\mathbf{z}}\,\frac{301.61}{\eta_2}\cos\left(2\pi\times10^9t-\frac{40\pi y}{3}\right)\\ &= \hat{\mathbf{z}}\,1.6\cos\left(2\pi\times10^9t-\frac{40\pi y}{3}\right)\quad(\mathrm{A/m}).\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{S}^{\mathrm{i}}_{\mathrm{av}} &= \hat{\mathbf{y}}\,\frac{|E_0|^2}{2\eta_1} = \hat{\mathbf{y}}\,\frac{(251.34)^2}{2\times125.67} = \hat{\mathbf{y}}\,251.34\quad(\mathrm{W/m^2}),\\ \mathbf{S}^{\mathrm{r}}_{\mathrm{av}} &= -\hat{\mathbf{y}}\,|\Gamma|^2(251.34) = \hat{\mathbf{y}}\,10.05\quad(\mathrm{W/m^2}),\\ \mathbf{S}^{\mathrm{t}}_{\mathrm{av}} &= \hat{\mathbf{y}}\,(251.34-10.05) = \hat{\mathbf{y}}\,241.29\quad(\mathrm{W/m^2}).\end{aligned}$$

Problem 8.4 A 200-MHz left-hand circularly polarized plane wave with an electric field modulus of 5 V/m is normally incident in air upon a dielectric medium with $\varepsilon_{\mathrm{r}} = 4$ and occupying the region defined by $z \geq 0$.

- **(a)** Write an expression for the electric field phasor of the incident wave, given that the field is a positive maximum at $z = 0$ and $t = 0$.
- **(b)** Calculate the reflection and transmission coefficients.
- **(c)** Write expressions for the electric field phasors of the reflected wave, the transmitted wave, and the total field in the region $z \leq 0$.

(d) Determine the percentages of the incident average power reflected by the boundary and transmitted into the second medium.

Solution:

(a)

$$k_1 = \frac{\omega}{c} = \frac{2\pi \times 2 \times 10^8}{3 \times 10^8} = \frac{4\pi}{3} \text{ rad/m},$$
$$k_2 = \frac{\omega}{u_{\mathrm{p}_2}} = \frac{\omega}{c}\sqrt{\varepsilon_{\mathrm{r}_2}} = \frac{4\pi}{3}\sqrt{4} = \frac{8\pi}{3} \text{ rad/m}.$$

LHC wave:

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = a_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}e^{j\pi/2})e^{-jkz} = a_0(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jkz},$$
$$\mathbf{E}^{\mathrm{i}}(z,t) = \hat{\mathbf{x}}a_0\cos(\omega t - kz) - \hat{\mathbf{y}}a_0\sin(\omega t - kz),$$
$$|\mathbf{E}^{\mathrm{i}}| = [a_0^2\cos^2(\omega t - kz) + a_0^2\sin^2(\omega t - kz)]^{1/2} = a_0 = 5 \quad \text{(V/m)}.$$

Hence,

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = 5(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-j4\pi z/3} \quad \text{(V/m)}.$$

(b)

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}}}} = \frac{\eta_0}{2} = 60\pi \quad (\Omega).$$

Equations (8.8a) and (8.9) give

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{60\pi - 120\pi}{60\pi + 120\pi} = \frac{-60}{180} = -\frac{1}{3}, \qquad \tau = 1 + \Gamma = \frac{2}{3}.$$

(c)

$$\widetilde{\mathbf{E}}^{\mathrm{r}} = 5\Gamma(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{jk_1z} = -\frac{5}{3}(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{j4\pi z/3} \quad \text{(V/m)},$$
$$\widetilde{\mathbf{E}}^{\mathrm{t}} = 5\tau(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jk_2z} = \frac{10}{3}(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-j8\pi z/3} \quad \text{(V/m)},$$
$$\widetilde{\mathbf{E}}_1 = \widetilde{\mathbf{E}}^{\mathrm{i}} + \widetilde{\mathbf{E}}^{\mathrm{r}} = 5(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\left[e^{-j4\pi z/3} - \frac{1}{3}e^{j4\pi z/3}\right] \quad \text{(V/m)}.$$

(d)

$$\text{\% of reflected power} = 100 \times |\Gamma|^2 = \frac{100}{9} = 11.11\%,$$
$$\text{\% of transmitted power} = 100 \times |\tau|^2\frac{\eta_1}{\eta_2} = 100 \times \left(\frac{2}{3}\right)^2 \times \frac{120\pi}{60\pi} = 88.89\%.$$

Problem 8.5 Repeat Problem 8.4 after replacing the dielectric medium with a poor conductor characterized by $\varepsilon_r = 2.25$, $\mu_r = 1$, and $\sigma = 10^{-4}$ S/m.

Solution:

(a) Medium 1:

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad k_1 = \frac{\omega}{c} = \frac{2\pi\times 2\times 10^8}{3\times 10^8} = \frac{4\pi}{3} \quad \text{(rad/m)}.$$

Medium 2:

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{10^{-4}\times 36\pi}{2\pi\times 2\times 10^8\times 2.25\times 10^{-9}} = 4\times 10^{-3}.$$

Hence, medium 2 is a low-loss dielectric. From Table 7-1,

$$\begin{aligned}
\alpha_2 &= \frac{\sigma_2}{2}\sqrt{\frac{\mu_2}{\varepsilon_2}} \\
&= \frac{\sigma_2}{2}\,\frac{120\pi}{\sqrt{\varepsilon_{r_2}}} = \frac{\sigma_2}{2}\times\frac{120\pi}{\sqrt{2.25}} = \frac{10^{-4}}{2}\times\frac{120\pi}{1.5} = 1.26\times 10^{-2} \quad \text{(NP/m)}, \\
\beta_2 &= \omega\sqrt{\mu_2\varepsilon_2} = \frac{\omega\sqrt{\varepsilon_{r_2}}}{c} = 2\pi \quad \text{(rad/m)}, \\
\eta_2 &= \sqrt{\frac{\mu_2}{\varepsilon_2}}\left(1+\frac{j\sigma_2}{2\omega\varepsilon_2}\right) = \frac{120\pi}{\sqrt{\varepsilon_{r_2}}}\left(1+j2\times 10^{-3}\right) \simeq \frac{120\pi}{1.5} = 80\pi \quad (\Omega).
\end{aligned}$$

LHC wave:

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{i}} &= a_0(\hat{\mathbf{x}}+j\hat{\mathbf{y}})e^{-jk_1z}, \\
|\widetilde{\mathbf{E}}^{\mathrm{i}}| &= a_0 = 5 \quad \text{(V/m)}, \\
\widetilde{\mathbf{E}}^{\mathrm{i}} &= 5(\hat{\mathbf{x}}+j\hat{\mathbf{y}})e^{-j4\pi z/3} \quad \text{(V/m)}.
\end{aligned}$$

(b) According to Eqs. (8.8a) and (8.9),

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{80\pi-120\pi}{80\pi+120\pi} = -0.2, \qquad \tau = 1+\Gamma = 1-0.2 = 0.8.$$

(c)

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{r}} &= 5\Gamma(\hat{\mathbf{x}}+j\hat{\mathbf{y}})e^{jk_1z} = -(\hat{\mathbf{x}}+j\hat{\mathbf{y}})e^{j4\pi z/3} \quad \text{(V/m)}, \\
\widetilde{\mathbf{E}}^{\mathrm{t}} &= 5\tau(\hat{\mathbf{x}}+j\hat{\mathbf{y}})e^{-\alpha_2 z}e^{-j\beta_z z} = 4(\hat{\mathbf{x}}+j\hat{\mathbf{y}})e^{-1.26\times 10^{-2}z}e^{-j2\pi z} \quad \text{(V/m)}, \\
\widetilde{\mathbf{E}}_1 &= \widetilde{\mathbf{E}}^{\mathrm{i}}+\widetilde{\mathbf{E}}^{\mathrm{r}} = 5(\hat{\mathbf{x}}+j\hat{\mathbf{y}})[e^{-j4\pi z/3}-0.2e^{j4\pi z/3}] \quad \text{(V/m)}.
\end{aligned}$$

(d)

$$\begin{aligned} \text{\% of reflected power} &= 100|\Gamma|^2 = 100(0.2)^2 = 4\%, \\ \text{\% of transmitted power} &= 100|\tau|^2 \frac{\eta_1}{\eta_2} = 100(0.8)^2 \times \frac{120\pi}{80\pi} = 96\%. \end{aligned}$$

Problem 8.6 A 50-MHz plane wave with electric field amplitude of 50 V/m is normally incident in air onto a semi-infinite, perfect dielectric medium with $\varepsilon_r = 36$. Determine (a) Γ, (b) the average power densities of the incident and reflected waves, and (c) the distance in the air medium from the boundary to the nearest minimum of the electric field intensity, $|\mathbf{E}|$.

Solution:

(a)

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \frac{120\pi}{\sqrt{\varepsilon_{r_2}}} = \frac{120\pi}{6} = 20\pi \quad (\Omega),$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{20\pi - 120\pi}{20\pi + 120\pi} = -0.71.$$

Hence, $|\Gamma| = 0.71$ and $\theta_\eta = 180°$.

(b)

$$\begin{aligned} S^{\text{i}}_{\text{av}} &= \frac{|E^{\text{i}}_0|^2}{2\eta_1} = \frac{(50)^2}{2 \times 120\pi} = 3.32 \quad (\text{W/m}^2), \\ S^{\text{r}}_{\text{av}} &= |\Gamma|^2 S^{\text{i}}_{\text{av}} = (0.71)^2 \times 3.32 = 1.67 \quad (\text{W/m}^2). \end{aligned}$$

(c) In medium 1 (air),

$$\lambda_1 = \frac{c}{f} = \frac{3 \times 10^8}{5 \times 10^7} = 6 \text{ m}.$$

From Eqs. (8.16) and (8.17),

$$\begin{aligned} l_{\max} &= \frac{\theta_r \lambda_1}{4\pi} = \frac{\pi \times 6}{4\pi} = 1.5 \text{ m}, \\ l_{\min} &= l_{\max} - \frac{\lambda_1}{4} = 1.5 - 1.5 = 0 \text{ m (at the boundary)}. \end{aligned}$$

Problem 8.7 What is the maximum amplitude of the total electric field in the air medium of Problem 8.6, and at what nearest distance from the boundary does it occur?

Solution: From Problem 8.6, $\Gamma = -0.71$ and $\lambda = 6$ m.

$$|\widetilde{\mathbf{E}}_1|_{\text{max}} = (1+|\Gamma|)E_0^i = (1+0.71)\times 50 = 85.5 \text{ V/m},$$
$$l_{\text{max}} = \frac{\theta_\text{r}\lambda_1}{4\pi} = \frac{\pi\times 6}{4\pi} = 1.5 \text{ m}.$$

Problem 8.8 Repeat Problem 8.6 after replacing the dielectric medium with a conductor with $\varepsilon_\text{r} = 1$, $\mu_\text{r} = 1$, and $\sigma = 2.78\times 10^{-3}$ S/m.

Solution:

(a) Medium 1:

$$\eta_1 = \eta_0 = 120\pi = 377 \quad (\Omega), \qquad \lambda_1 = \frac{c}{f} = \frac{3\times 10^8}{5\times 10^7} = 6 \text{ m},$$

Medium 2:

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{2.78\times 10^{-3}\times 36\pi}{2\pi\times 5\times 10^7\times 10^{-9}} = 1.$$

Hence, Medium 2 is a quasi-conductor. From Eq. (7.70),

$$\begin{aligned}\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}\left(1-j\frac{\varepsilon_2''}{\varepsilon_2'}\right)^{-1/2} &= 120\pi\left(1-j\frac{\sigma_2}{\omega\varepsilon_2}\right)^{-1/2}\\ &= 120\pi(1-j1)^{-1/2}\\ &= 120\pi(\sqrt{2})^{-1/2}e^{j22.5^\circ} = (292.88+j121.31) \quad (\Omega).\end{aligned}$$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{(292.88+j121.31)-377}{(292.88+j121.31)+377} = -0.09+j0.12 = 0.22\angle 114.5^\circ.$$

(b)

$$S_{\text{av}}^{\text{i}} = \frac{|E_0^{\text{i}}|^2}{2\eta_1} = \frac{50^2}{2\times 120\pi} = 3.32 \quad (\text{W/m}^2),$$
$$|S_{\text{av}}^{\text{r}}| = |\Gamma|^2 S_{\text{av}}^{\text{i}} = (0.22)^2(3.32) = 0.16 \quad (\text{W/m}^2).$$

(c) In medium 1 (air),

$$\lambda_1 = \frac{c}{f} = \frac{3\times 10^8}{5\times 10^7} = 6 \text{ m}.$$

For $\theta_\text{r} = 114.5^\circ = 2$ rad, Eqs. (8.16) and (8.17) give

$$l_{\text{max}} = \frac{\theta_\text{r}\lambda_1}{4\pi} + \frac{(0)\lambda_1}{2} = \frac{2(6)}{4} + 0 = 3 \text{ m},$$

$$l_{\min} = l_{\max} - \frac{\lambda_1}{4} = 3 - \frac{6}{4} = 3 - 1.5 = 1.5 \text{ m.}$$

Problem 8.9 The three regions shown in Fig. 8-32 (P8.9) contain perfect dielectrics. For a wave in medium 1 incident normally upon the boundary at $z = -d$, what combination of ε_{r_2} and d produce no reflection? Express your answers in terms of ε_{r_1}, ε_{r_3} and the oscillation frequency of the wave, f.

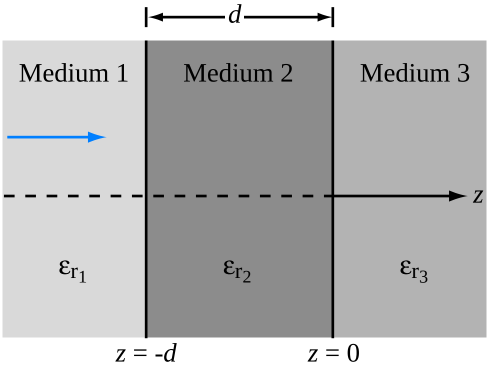


Figure P8.9: Three dielectric regions.

Solution: By analogy with the transmission-line case, there will be no reflection at $z = -d$ if medium 2 acts as a quarter-wave transformer, which requires that

$$d = \frac{\lambda_2}{4}$$

and

$$\eta_2 = \sqrt{\eta_1 \eta_3}\,.$$

The second condition may be rewritten as

$$\frac{\eta_0}{\sqrt{\varepsilon_{r_2}}} = \left[\frac{\eta_0}{\sqrt{\varepsilon_{r_1}}} \frac{\eta_0}{\sqrt{\varepsilon_{r_3}}} \right]^{1/2}, \qquad \text{or} \quad \varepsilon_{r_2} = \sqrt{\varepsilon_{r_1}\varepsilon_{r_3}}\,,$$

$$\lambda_2 = \frac{\lambda_0}{\sqrt{\varepsilon_{r_2}}} = \frac{c}{f\sqrt{\varepsilon_{r_2}}} = \frac{c}{f(\varepsilon_{r_1}\varepsilon_{r_3})^{1/4}}\,,$$

and

$$d = \frac{c}{4f(\varepsilon_{r_1}\varepsilon_{r_3})^{1/4}}\,.$$

Problem 8.10 For the configuration shown in Fig. 8-32 (P8.9), use transmission-line equations (or the Smith chart) to calculate the input impedance at $z = -d$ for $\varepsilon_{r_1} = 1$, $\varepsilon_{r_2} = 9$, $\varepsilon_{r_3} = 4$, $d = 1.2$ m, and $f = 50$ MHz. Also determine the fraction of the incident average power density reflected by the structure. Assume all media are lossless and nonmagnetic.

Solution: In medium 2,

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_{r_2}}} = \frac{c}{f\sqrt{\varepsilon_{r_2}}} = \frac{3\times 10^8}{5\times 10^7\times 3} = 2 \text{ m}.$$

Hence,

$$\beta_2 = \frac{2\pi}{\lambda_2} = \pi \text{ rad/m}, \qquad \beta_2 d = 1.2\pi \text{ rad}.$$

At $z = -d$, the input impedance of a transmission line with load impedance Z_L is given by Eq. (2.63) as

$$Z_{\text{in}}(-d) = Z_0\left(\frac{Z_L + jZ_0\tan\beta_2 d}{Z_0 + jZ_L\tan\beta_2 d}\right).$$

In the present case, $Z_0 = \eta_2 = \eta_0/\sqrt{\varepsilon_{r_2}} = \eta_0/3$ and $Z_L = \eta_3 = \eta_0/\sqrt{\varepsilon_{r_3}} = \eta_0/2$, where $\eta_0 = 120\pi$ (Ω). Hence,

$$Z_{\text{in}}(-d) = \eta_2\left(\frac{\eta_3 + j\eta_2\tan\beta_2 d}{\eta_2 + j\eta_3\tan\beta_2 d}\right) = \frac{\eta_0}{3}\left(\frac{\frac{1}{2} + j\left(\frac{1}{3}\right)\tan 1.2\pi}{\frac{1}{3} + j\left(\frac{1}{2}\right)\tan 1.2\pi}\right) = \eta_0(0.35 - j0.14).$$

At $z = -d$,

$$\Gamma = \frac{Z_{\text{in}} - Z_1}{Z_{\text{in}} + Z_1} = \frac{\eta_0(0.35 - j0.14) - \eta_0}{\eta_0(0.35 - j0.14) + \eta_0} = 0.49e^{-j162.14^\circ}.$$

Fraction of incident power reflected by the structure is $|\Gamma|^2 = |0.49|^2 = 0.24$.

Problem 8.11 Repeat Problem 8.10 after interchanging ε_{r_1} and ε_{r_3}.

Solution: In medium 2,

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_{r_2}}} = \frac{c}{f\sqrt{\varepsilon_{r_2}}} = \frac{3\times 10^8}{5\times 10^7\times 3} = 2 \text{ m}.$$

Hence,

$$\beta_2 = \frac{2\pi}{\lambda_2} = \pi \text{ rad/m}, \qquad \beta_2 d = 1.2\pi \text{ rad}.$$

At $z = -d$, the input impedance of a transmission line with impedance Z_L is given as Eq. (2.63),

$$Z_{\text{in}}(-d) = Z_0 \left(\frac{Z_L + jZ_0 \tan\beta d}{Z_0 + jZ_L \tan\beta 2d} \right).$$

In the present case, $Z_0 = \eta_2 = \eta_0/\sqrt{\varepsilon_{r_2}} = \eta_0/3$, $Z_L = \eta_3 = \eta_0/\sqrt{\varepsilon_{r_1}} = \eta_0$, where $\eta_0 = 120\pi$ (Ω). Hence,

$$\begin{aligned} Z_{\text{in}}(-d) &= \eta_2 \left(\frac{\eta_3 + j\eta_2 \tan 1.2\pi}{\eta_2 + j\eta_3 \tan 1.2\pi} \right) \\ &= \frac{\eta_0}{3} \left(\frac{1 + (j/3)\tan 1.2\pi}{(1/3) + j\tan 1.2\pi} \right) \\ &= \eta_0 \left(\frac{1 + (j/3)\tan 1.2\pi}{1 + j3\tan 1.2\pi} \right) = (0.266 - j0.337)\eta_0 = 0.43\eta_0 \angle{-51.7^\circ}. \end{aligned}$$

At $z = -d$,

$$\Gamma = \frac{Z_{\text{in}} - Z_1}{Z_{\text{in}} + Z_1} = \frac{0.43\angle{-51.7^\circ} - \frac{1}{2}}{0.43\angle{-51.7^\circ} + \frac{1}{2}} = 0.49\angle{-101.1^\circ}.$$

Fraction of incident power reflected by structure is $|\Gamma|^2 = 0.24$.

Problem 8.12 Orange light of wavelength 0.61 μm in air enters a block of glass with $\varepsilon_r = 1.44$. What color would it appear to a sensor embedded in the glass? The wavelength ranges of colors are violet (0.39 to 0.45 μm), blue (0.45 to 0.49 μm), green (0.49 to 0.58 μm), yellow (0.58 to 0.60 μm), orange (0.60 to 0.62 μm), and red (0.62 to 0.78 μm).

Solution: In the glass,

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{0.61}{\sqrt{1.44}} = 0.508 \ \mu\text{m}.$$

The light would appear green.

Problem 8.13 A plane wave of unknown frequency is normally incident in air upon the surface of a perfect conductor. Using an electric-field meter, it was determined that the total electric field in the air medium is always zero when measured at a distance of 2 m from the conductor surface. Moreover, no such nulls were observed at distances closer to the conductor. What is the frequency of the incident wave?

Solution: The electric field of the standing wave is zero at the conductor surface, and the standing wave pattern repeats itself every $\lambda/2$. Hence,

$$\frac{\lambda}{2} = 2 \text{ m}, \qquad \text{or } \lambda = 4 \text{ m},$$

in which case

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{4} = 7.5 \times 10^7 = 75 \text{ MHz}.$$

Problem 8.14 Consider a thin film of soap in air under illumination by yellow light with $\lambda = 0.6$ μm in vacuum. If the film is treated as a planar dielectric slab with $\varepsilon_r = 1.72$, surrounded on both sides by air, what film thickness would produce strong reflection of the yellow light at normal incidence?

Solution: The transmission line analogue of the soap-bubble wave problem is shown in Fig. P8.14(b) where the load Z_L is equal to η_0, the impedance of the air medium on the other side of the bubble. That is,

$$\eta_0 = 377 \ \Omega, \qquad \eta_1 = \frac{377}{\sqrt{1.72}} = 287.5 \ \Omega.$$

The normalized load impedance is

$$z_L = \frac{\eta_0}{\eta_1} = 1.31.$$

For the reflection by the soap bubble to be the largest, Z_{in} needs to be the most different from η_0. This happens when z_L is transformed through a length $\lambda/4$. Hence,

$$L = \frac{\lambda}{4} = \frac{\lambda_0}{4\sqrt{\varepsilon_r}} = \frac{0.6 \ \mu\text{m}}{4\sqrt{1.72}} = 0.115 \ \mu\text{m},$$

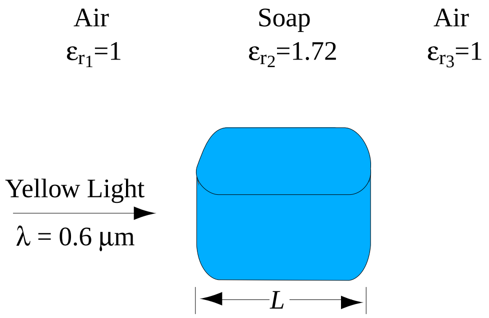
where λ is the wavelength of the soap bubble material. Strong reflections will also occur if the thickness is greater than L by integer multiples of $n\lambda/2 = (0.23n)$ μm.

Hence, in general

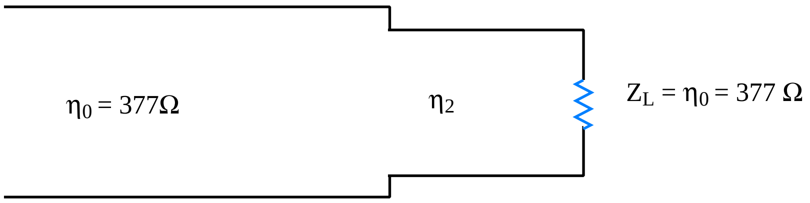
$$L = (0.115 + 0.23n) \ \mu\text{m}, \qquad n = 0, 1, 2, \dots .$$

According to Section 2-7.5, transforming a load $Z_L = 377$ Ω through a $\lambda/4$ section of $Z_0 = 287.5$ Ω ends up presenting an input impedance of

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{(287.5)^2}{377} = 219.25 \ \Omega.$$



(a) Yellow light incident on soap bubble.



(b) Transmission-line equivalent circuit

Figure P8.14: Diagrams for Problem 8.14.

This Z_{in} is at the input side of the soap bubble. The reflection coefficient at that interface is

$$\Gamma = \frac{Z_{\text{in}} - \eta_0}{Z_{\text{in}} + \eta_0} = \frac{219.25 - 377}{219.25 + 377} = -0.27.$$

Any other thickness would produce a reflection coeffficient with a smaller magnitude.

Problem 8.15 A 5-MHz plane wave with electric field amplitude of 10 (V/m) is normally incident in air onto the plane surface of a semi-infinite conducting material with $\varepsilon_r = 4$, $\mu_r = 1$, and $\sigma = 100$ (S/m). Determine the average power dissipated (lost) per unit cross-sectional area in a 2-mm penetration of the conducting medium.

Solution: For convenience, let us choose $\mathbf{E}^i$ to be along $\hat{\mathbf{x}}$ and the incident direction to be $+\hat{\mathbf{z}}$. With

$$k_1 = \frac{\omega}{c} = \frac{2\pi \times 5 \times 10^6}{3 \times 10^8} = \frac{\pi}{30} \quad \text{(rad/m)},$$

we have

$$\begin{aligned}
\mathbf{E}^{\mathrm{i}} &= \hat{\mathbf{x}}\,10\cos\left(\pi\times 10^7 t - \frac{\pi}{30}z\right) \quad \text{(V/m)},\\
\eta_1 &= \eta_0 = 377\ \Omega.
\end{aligned}$$

From Table 7-1,

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon_{\mathrm{r}}\varepsilon_0} = \frac{100\times 36\pi}{\pi\times 10^7\times 4\times 10^{-9}} = 9\times 10^4,$$

which makes the material a good conductor, for which

$$\begin{aligned}
\alpha_2 &= \sqrt{\pi f\mu\sigma} = \sqrt{\pi\times 5\times 10^6\times 4\pi\times 10^{-7}\times 100} = 44.43 \quad \text{(Np/m)},\\
\beta_2 &= 44.43 \quad \text{(rad/m)},\\
\eta_{\mathrm{c}_2} &= (1+j)\,\frac{\alpha_2}{\sigma} = (1+j)\,\frac{44.43}{100} = 0.44\,(1+j)\ \Omega.
\end{aligned}$$

According to the expression for $\mathbf{S}_{\mathrm{av}_2}$ given in the answer to Exercise 8.3,

$$\mathbf{S}_{\mathrm{av}_2} = \hat{\mathbf{z}}\,|\tau|^2\,\frac{|E_0^{\mathrm{i}}|^2}{2}\,e^{-2\alpha_2 z}\,\mathfrak{Re}\left(\frac{1}{\eta^*_{\mathrm{c}_2}}\right).$$

The power lost is equal to the difference between $\mathbf{S}_{\mathrm{av}_2}$ at $z=0$ and $\mathbf{S}_{\mathrm{av}_2}$ at $z=2$ mm. Thus,

$$\begin{aligned}
P' &= \text{power lost per unit cross-sectional area}\\
&= S_{\mathrm{av}_2}(0) - S_{\mathrm{av}_2}(z=2\text{ mm})\\
&= |\tau|^2\,\frac{|E_0^{\mathrm{i}}|^2}{2}\,\mathfrak{Re}\left(\frac{1}{\eta^*_{\mathrm{c}_2}}\right)\,[1-e^{-2\alpha_2 z_1}]
\end{aligned}$$

where $z_1 = 2$ mm.

$$\begin{aligned}
\tau &= 1+\Gamma\\
&= 1+\frac{\eta_2-\eta_1}{\eta_2+\eta_1} = 1+\frac{0.44\,(1+j)-377}{0.44\,(1+j)+377} \approx 0.0023\,(1+j) = 3.3\times 10^{-3}e^{j45^\circ}.\\
\mathfrak{Re}\left(\frac{1}{\eta^*_{\mathrm{c}_2}}\right) &= \mathfrak{Re}\left(\frac{1}{0.44\,(1+j)^*}\right)\\
&= \mathfrak{Re}\left(\frac{1}{0.44\,(1-j)}\right) = \mathfrak{Re}\left(\frac{1+j}{0.44\times 2}\right) = \frac{1}{0.88} = 1.14,\\
P' &= (3.3\times 10^{-3})^2\,\frac{10^2}{2}\times 1.14\,[1-e^{-2\times 44.43\times 2\times 10^{-3}}] = 1.01\times 10^{-4} \quad \text{(W/m}^2\text{)}.
\end{aligned}$$

Problem 8.16 A 0.5-MHz antenna carried by an airplane flying over the ocean surface generates a wave that approaches the water surface in the form of a normally incident plane wave with an electric-field amplitude of 3,000 (V/m). Sea water is characterized by $\varepsilon_r = 72$, $\mu_r = 1$, and $\sigma = 4$ (S/m). The plane is trying to communicate a message to a submarine submerged at a depth d below the water surface. If the submarine's receiver requires a minimum signal amplitude of 0.01 (μV/m), what is the maximum depth d to which successful communication is still possible?

Solution: For sea water at 0.5 MHz,

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{4 \times 36\pi}{2\pi \times 0.5 \times 10^6 \times 72 \times 10^{-9}} = 2000.$$

Hence, sea water is a good conductor, in which case we use the following expressions from Table 7-1:

$$\begin{aligned}
\alpha_2 &= \sqrt{\pi f \mu\sigma} = \sqrt{\pi \times 0.5 \times 10^6 \times 4\pi \times 10^{-7} \times 4} = 2.81 \quad \text{(Np/m)},\\
\beta_2 &= 2.81 \quad \text{(rad/m)},\\
\eta_{c_2} &= (1+j)\,\frac{\alpha_2}{\sigma} = (1+j)\,\frac{2.81}{4} = 0.7\,(1+j)\ \Omega,\\
\Gamma &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{0.7\,(1+j) - 377}{0.7\,(1+j) + 377} = (-0.9963 + j3.7 \times 10^{-3}),\\
\tau &= 1 + \Gamma = 5.24 \times 10^{-3} e^{j44.89^\circ},\\
|E^{\mathrm{t}}| &= |\tau E_0^{\mathrm{i}} e^{-\alpha_2 d}|.
\end{aligned}$$

We need to find the depth z at which $|E^{\mathrm{t}}| = 0.01\ \mu\text{V/m} = 10^{-8}$ V/m.

$$\begin{aligned}
10^{-8} &= 5.24 \times 10^{-3} \times 3 \times 10^3 e^{-2.81d},\\
e^{-2.81d} &= 6.36 \times 10^{-10},\\
-2.81d &= \ln(6.36 \times 10^{-10}) = -21.18,
\end{aligned}$$

or

$$d = 7.54 \quad \text{(m)}.$$

Sections 8-2 and 8-3: Snell's Laws and Fiber Optics

Problem 8.17 A light ray is incident on a prism at an angle θ as shown in Fig. 8-33 (P8.17). The ray is refracted at the first surface and again at the second surface. In terms of the apex angle ϕ of the prism and its index of refraction n, determine the smallest value of θ for which the ray will emerge from the other side. Find this minimum θ for $n = 4$ and $\phi = 60°$.

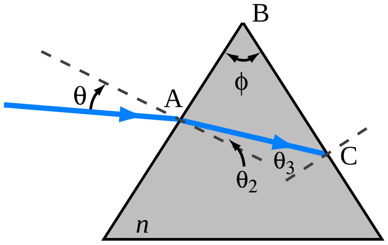


Figure P8.17: Prism of Problem 8.17.

Solution: For the beam to emerge at the second boundary, it is necessary that

$$\theta_3 < \theta_c,$$

where $\sin\theta_c = 1/n$. From the geometry of triangle ABC,

$$180° = \phi + (90° - \theta_2) + (90° - \theta_3),$$

or $\theta_2 = \phi - \theta_3$. At the first boundary, $\sin\theta = n\sin\theta_2$. Hence,

$$\sin\theta_{\text{min}} = n\sin(\phi - \theta_3) = n\sin\left(\phi - \sin^{-1}\left(\frac{1}{n}\right)\right),$$

or

$$\theta_{\text{min}} = \sin^{-1}\left[n\sin\left(\phi - \sin^{-1}\left(\frac{1}{n}\right)\right)\right].$$

For $n = 4$ and $\phi = 60°$,

$$\theta_{\text{min}} = \sin^{-1}\left[4\sin(60° - \sin^{-1}\left(\frac{1}{4}\right)\right] = 20.4°.$$

Problem 8.18 For some types of glass, the index of refraction varies with wavelength. A prism made of a material with

$$n = 1.71 - \frac{4}{30}\lambda_0, \quad (\lambda_0 \text{ in } \mu\text{m}),$$

where λ_0 is the wavelength in vacuum, was used to disperse white light as shown in Fig. 8-34 (P8.18). The white light is incident at an angle of 50°, the wavelength λ_0 of red light is 0.7 μm and that of violet light is 0.4 μm. Determine the angular dispersion in degrees.

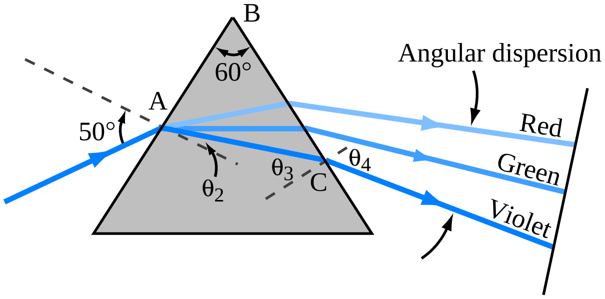


Figure P8.18: Prism of Problem 8.18.

Solution: For violet,

$$n_v = 1.71 - \frac{4}{30} \times 0.4 = 1.66, \qquad \sin\theta_2 = \frac{\sin\theta}{n_v} = \frac{\sin 50^\circ}{1.66},$$

or

$$\theta_2 = 27.48^\circ.$$

From the geometry of triangle *ABC*,

$$180^\circ = 60^\circ + (90^\circ - \theta_2) + (90^\circ - \theta_3),$$

or

$$\theta_3 = 60^\circ - \theta_2 = 60 - 27.48^\circ = 32.52^\circ,$$

and

$$\sin\theta_4 = n_v \sin\theta_3 = 1.66 \sin 32.52^\circ = 0.89,$$

or

$$\theta_4 = 63.18^\circ.$$

For red,

$$\begin{aligned}
n_{\mathrm{r}} &= 1.71 - \frac{4}{30} \times 0.7 = 1.62, \\
\theta_2 &= \sin^{-1}\left[\frac{\sin 50^\circ}{1.62}\right] = 28.22^\circ, \\
\theta_3 &= 60^\circ - 28.22^\circ = 31.78^\circ, \\
\theta_4 &= \sin^{-1}[1.62 \sin 31.78^\circ] = 58.56^\circ.
\end{aligned}$$

Hence, angular dispersion $= 63.18^\circ - 58.56^\circ = 4.62^\circ$.

Problem 8.19 The two prisms in Fig. 8-35 (P8.19) are made of glass with $n = 1.5$. What fraction of the power density carried by the ray incident upon the top prism emerges from bottom prism? Neglect multiple internal reflections.

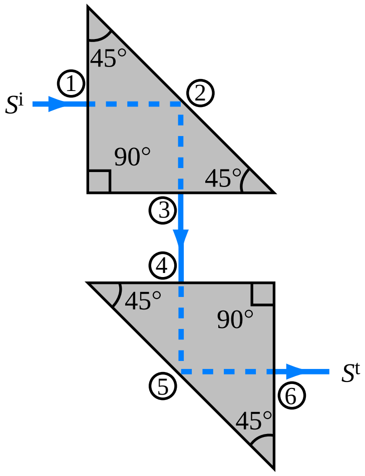


Figure P8.19: Periscope problem.

Solution: Using $\eta = \eta_0/n$, at interfaces 1 and 4,

$$\Gamma_{\mathrm{a}} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.5}{1 + 1.5} = -0.2.$$

At interfaces 3 and 6,

$$\Gamma_{\mathrm{b}} = -\Gamma_{\mathrm{a}} = 0.2.$$

At interfaces 2 and 5,

$$\theta_c = \sin^{-1}\left(\frac{1}{n}\right) = \sin^{-1}\left(\frac{1}{1.5}\right) = 41.81^\circ.$$

Hence, total internal reflection takes place at those interfaces. At interfaces 1, 3, 4 and 6, the ratio of power density transmitted to that incident is $(1-\Gamma^2)$. Hence,

$$\frac{S^t}{S^i} = (1-\Gamma^2)^4 = (1-(0.2)^2)^4 = 0.85.$$

Problem 8.20 A light ray incident at 45° passes through two dielectric materials with the indices of refraction and thicknesses given in Fig. 8-36 (P8.20). If the ray strikes the surface of the first dielectric at a height of 2 cm, at what height will it strike the screen?

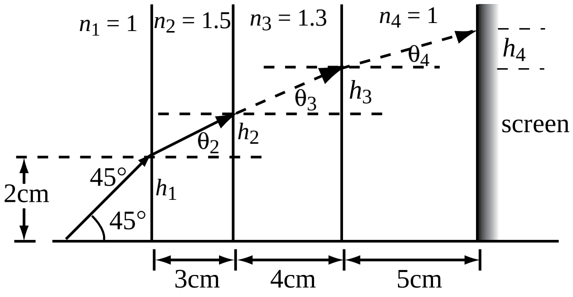


Figure P8.20: Light incident on a screen through a multi-layered dielectric (Problem 8.20).

Solution:

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 = \frac{1}{1.5}\sin 45^\circ = 0.47.$$

Hence,

$$\theta_2 = 28.13^\circ,$$
$$h_2 = 3\text{ cm} \times \tan\theta_2 = 3\text{ cm} \times 0.53 = 1.6\text{ cm},$$
$$\sin\theta_3 = \frac{n_2}{n_3}\sin\theta_2 = \frac{1.5}{1.3}\sin 28.13^\circ = 0.54.$$

Hence,

$$\theta_3 = 32.96^\circ,$$

$$h_3 = 4 \text{ cm} \times \tan 32.96^\circ = 2.6 \text{ cm},$$
$$\sin\theta_4 = \frac{n_3}{n_4}\sin\theta_3 = 0.707.$$

Hence,

$$\theta_4 = 45^\circ,$$
$$h_4 = 5 \text{ cm} \times \tan 45^\circ = 5 \text{ cm}.$$

Total height $= h_1 + h_2 + h_3 + h_4 = (2 + 1.6 + 2.6 + 5) = 11.2$ cm.

Problem 8.21 Figure P8.21 depicts a beaker containing a block of glass on the bottom and water over it. The glass block contains a small air bubble at an unknown depth below the water surface. When viewed from above at an angle of 60°, the air bubble appears at a depth of 6.81 cm. What is the true depth of the air bubble?

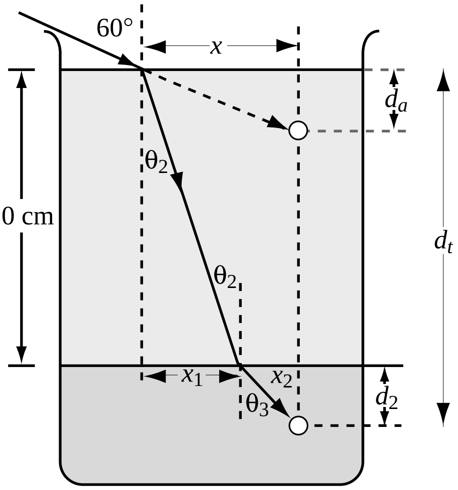


Figure P8.21: Apparent position of the air bubble in Problem 8.21.

Solution: Let

$$d_\text{a} = 6.81 \text{ cm} = \text{apparent depth},$$
$$d_\text{t} = \text{true depth}.$$

$$\theta_2 = \sin^{-1}\left[\frac{n_1}{n_2}\sin\theta_\text{i}\right] = \sin^{-1}\left[\frac{1}{1.33}\sin 60^\circ\right] = 40.6^\circ,$$

$$\theta_3 = \sin^{-1}\left[\frac{n_1}{n_3}\sin\theta_i\right] = \sin^{-1}\left[\frac{1}{1.6}\sin 60^\circ\right] = 32.77^\circ,$$
$$x_1 = (10\text{ cm}) \times \tan 40.6^\circ = 8.58\text{ cm},$$
$$x = d_a \cot 30^\circ = 6.81 \cot 30^\circ = 11.8\text{ cm}.$$

Hence,

$$x_2 = x - x_1 = 11.8 - 8.58 = 3.22\text{ cm},$$

and

$$d_2 = x_2 \cot 32.77^\circ = (3.22\text{ cm}) \times \cot 32.77^\circ = 5\text{ cm}.$$

Hence, $d_t = (10+5) = 15$ cm.

Problem 8.22 A glass semicylinder with $n = 1.5$ is positioned such that its flat face is horizontal, as shown in Fig. 8-38 (P8.22). Its horizontal surface supports a drop of oil, as shown. When light is directed radially toward the oil, total internal reflection occurs if θ exceeds 53°. What is the index of refraction of the oil?

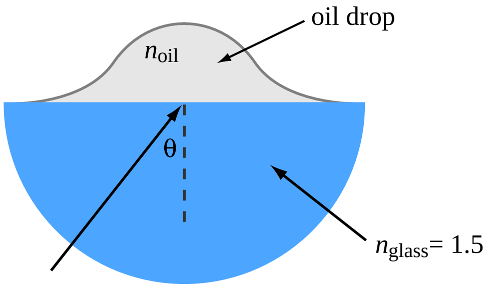


Figure P8.22: Oil drop on the flat surface of a glass semicylinder (Problem 8.22).

Solution:

$$\sin\theta_c = \frac{n_2}{n_1} = \frac{n_{\text{oil}}}{1.5},$$
$$n_{\text{oil}} = 1.5 \sin 53^\circ = 1.2.$$

Problem 8.23 A penny lies at the bottom of a water fountain at a depth of 30 cm. Determine the diameter of a piece of paper which, if placed to float on the surface of

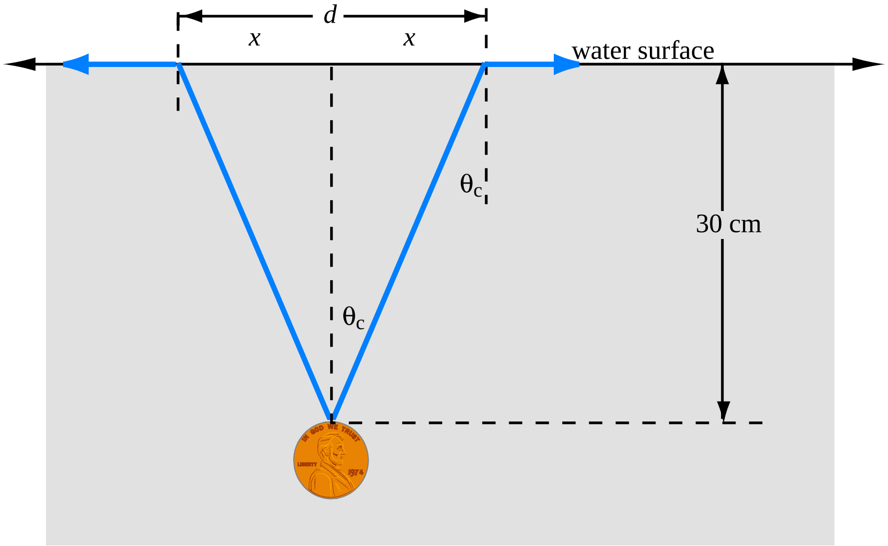


Figure P8.23: Light cone bounded by total internal reflection.

the water directly above the penny, would totally obscure the penny from view. Treat the penny as a point and assume that $n = 1.33$ for water.

Solution:

$$\theta_c = \sin^{-1}\left[\frac{1}{1.33}\right] = 48.75^\circ,$$

$$d = 2x = 2[(30 \text{ cm})\tan\theta_c] = (60 \text{ cm}) \times \tan 48.75^\circ = 68.42 \text{ cm}.$$

Problem 8.24 Suppose the optical fiber of Example 8-5 is submerged in water (with $n = 1.33$) instead of air. Determine θ_a and f_p in that case.

Solution: With $n_0 = 1.33$, $n_f = 1.52$ and $n_c = 1.49$, Eq. (8.40) gives

$$\sin\theta_a = \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2} = \frac{1}{1.33}\left[(1.52)^2 - (1.49)^2\right]^{1/2} = 0.23,$$

or

$$\theta_a = 13.1^\circ.$$

The data rate f_p given by Eq. (8.45) is not a function of n_0, and therefore it remains unchanged at 4.9 (Mb/s).

Problem 8.25 Equation (8.45) was derived for the case where the light incident upon the sending end of the optical fiber extends over the entire acceptance cone shown in Fig. 8-12(b). Suppose the incident light is constrained to a narrower range extending between normal incidence and θ', where $\theta' < \theta_a$.

(a) Obtain an expression for the maximum data rate f_p in terms of θ'.

(b) Evaluate f_p for the fiber of Example 8-5 when $\theta' = 5°$.

Solution:

(a) For $\theta_i = \theta'$,

$$\sin\theta_2 = \frac{1}{n_f}\sin\theta',$$

$$l_{\max} = \frac{l}{\cos\theta_2} = \frac{l}{\sqrt{1-\sin^2\theta_2}} = \frac{l}{\sqrt{1-\left(\frac{\sin\theta'}{n_f}\right)^2}} = \frac{ln_f}{\sqrt{n_f^2-(\sin\theta')^2}},$$

$$t_{\max} = \frac{l_{\max}}{u_p} = \frac{l_{\max}n_f}{c} = \frac{ln_f^2}{c\sqrt{n_f^2-(\sin\theta')^2}},$$

$$t_{\min} = \frac{l}{u_p} = l\,\frac{n_f}{c},$$

$$\tau = \Delta t = t_{\max} - t_{\min} = l\,\frac{n_f}{c}\left[\frac{n_f}{\sqrt{n_f^2-(\sin\theta')^2}} - 1\right],$$

$$f_p = \frac{1}{2\tau} = \frac{c}{2ln_f}\left[\frac{n_f}{\sqrt{n_f^2-(\sin\theta')^2}} - 1\right]^{-1} \quad \text{(bits/s)}.$$

(b) For:

$$n_f = 1.52,$$
$$\theta' = 5°,$$
$$l = 1 \text{ km},$$
$$c = 3\times10^8 \text{ m/s},$$
$$f_p = 59.88 \quad \text{(Mb/s)}.$$

Sections 8-4 and 8-5: Reflection and Transmission at Oblique Incidence

Problem 8.26 A plane wave in air with

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 20 e^{-j(3x+4z)} \quad \text{(V/m)},$$

is incident upon the planar surface of a dielectric material, with $\varepsilon_{\mathrm{r}} = 4$, occupying the half space $z \geq 0$. Determine:

- **(a)** the polarization of the incident wave,
- **(b)** the angle of incidence,
- **(c)** the time-domain expressions for the reflected electric and magnetic fields,
- **(d)** the time-domain expressions for the transmitted electric and magnetic fields, and
- **(e)** the average power density carried by the wave in the dielectric medium.

Solution:

(a) $\widetilde{\mathbf{E}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 20 e^{-j(3x+4z)}$ V/m.

Since $\mathbf{E}^{\mathrm{i}}$ is along $\hat{\mathbf{y}}$, which is perpendicular to the plane of incidence, the wave is perpendicularly polarized.

(b) From Eq. (8.48a), the argument of the exponential is

$$-jk_1(x\sin\theta_{\mathrm{i}} + z\cos\theta_{\mathrm{i}}) = -j(3x+4z).$$

Hence,

$$k_1\sin\theta_{\mathrm{i}} = 3, \qquad k_1\cos\theta_{\mathrm{i}} = 4,$$

from which we determine that

$$\tan\theta_{\mathrm{i}} = \frac{3}{4} \qquad \text{or} \qquad \theta_{\mathrm{i}} = 36.87^\circ,$$

and

$$k_1 = \sqrt{3^2+4^2} = 5 \quad \text{(rad/m)}.$$

Also,

$$\omega = u_{\mathrm{p}}k = ck = 3\times 10^8 \times 5 = 1.5\times 10^9 \quad \text{(rad/s)}.$$

(c)

$$\eta_1 = \eta_0 = 377\ \Omega,$$

$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_2}}} = \frac{\eta_0}{2} = 188.5\ \Omega,$$

$$\theta_{\mathrm{t}} = \sin^{-1}\left[\frac{\sin\theta_{\mathrm{i}}}{\sqrt{\varepsilon_{\mathrm{r}_2}}}\right] = \sin^{-1}\left[\frac{\sin 36.87^\circ}{\sqrt{4}}\right] = 17.46^\circ,$$

$$\Gamma_\perp = \frac{\eta_2\cos\theta_\mathrm{i} - \eta_1\cos\theta_\mathrm{t}}{\eta_2\cos\theta_\mathrm{i} + \eta_1\cos\theta_\mathrm{t}} = -0.41,$$
$$\tau_\perp = 1 + \Gamma_\perp = 0.59.$$

In accordance with Eq. (8.49a), and using the relation $E_0^\mathrm{r} = \Gamma_\perp E_0^\mathrm{i}$,

$$\widetilde{\mathbf{E}}^\mathrm{r} = -\hat{\mathbf{y}}\, 8.2\, e^{-j(3x-4z)},$$
$$\widetilde{\mathbf{H}}^\mathrm{r} = (\hat{\mathbf{x}}\cos\theta_\mathrm{i} + \hat{\mathbf{z}}\sin\theta_\mathrm{i})\,\frac{8.2}{\eta_0}\, e^{-j(3x-4z)},$$

where we used the fact that $\theta_\mathrm{i} = \theta_\mathrm{r}$ and the z-direction has been reversed.

$$\mathbf{E}^\mathrm{r} = \mathfrak{Re}[\widetilde{\mathbf{E}}^\mathrm{r} e^{j\omega t}] = -\hat{\mathbf{y}}\, 8.2\cos(1.5\times 10^9 t - 3x + 4z) \quad \text{(V/m)},$$
$$\mathbf{H}^\mathrm{r} = (\hat{\mathbf{x}}\, 17.4 + \hat{\mathbf{z}}\, 13.06)\cos(1.5\times 10^9 t - 3x + 4z) \quad \text{(mA/m)}.$$

(d) In medium 2,

$$k_2 = k_1\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 5\sqrt{4} = 20 \quad \text{(rad/m)},$$

and

$$\theta_\mathrm{t} = \sin^{-1}\left[\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_\mathrm{i}\right] = \sin^{-1}\left[\frac{1}{2}\sin 36.87^\circ\right] = 17.46^\circ$$

and the exponent of $\mathbf{E}^\mathrm{t}$ and $\mathbf{H}^\mathrm{t}$ is

$$-jk_2(x\sin\theta_\mathrm{t} + z\cos\theta_\mathrm{t}) = -j10(x\sin 17.46^\circ + z\cos 17.46^\circ) = -j(3x + 9.54z).$$

Hence,

$$\widetilde{\mathbf{E}}^\mathrm{t} = \hat{\mathbf{y}}\, 20\times 0.59\, e^{-j(3x+9.54z)},$$
$$\widetilde{\mathbf{H}}^\mathrm{t} = (-\hat{\mathbf{x}}\cos\theta_\mathrm{t} + \hat{\mathbf{z}}\sin\theta_\mathrm{t})\,\frac{20\times 0.59}{\eta_2}\, e^{-j(3x+9.54z)}.$$
$$\mathbf{E}^\mathrm{t} = \mathfrak{Re}[\widetilde{\mathbf{E}}^\mathrm{t} e^{j\omega t}] = \hat{\mathbf{y}}\, 11.8\cos(1.5\times 10^9 t - 3x - 9.54z) \quad \text{(V/m)},$$
$$\mathbf{H}^\mathrm{t} = (-\hat{\mathbf{x}}\cos 17.46^\circ + \hat{\mathbf{z}}\sin 17.46^\circ)\,\frac{11.8}{188.5}\cos(1.5\times 10^9 t - 3x - 9.54z)$$
$$= (-\hat{\mathbf{x}}\, 59.72 + \hat{\mathbf{z}}\, 18.78)\cos(1.5\times 10^9 t - 3x - 9.54z) \quad \text{(mA/m)}.$$

(e)

$$S_\mathrm{av}^\mathrm{t} = \frac{|E_0^\mathrm{t}|^2}{2\eta_2} = \frac{(11.8)^2}{2\times 188.5} = 0.36 \quad \text{(W/m}^2\text{)}.$$

Problem 8.27 Repeat Problem 8.26 for a wave in air with

$$\widetilde{\mathbf{H}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 2 \times 10^{-2} e^{-j(8x+6z)} \quad \text{(A/m)},$$

incident upon the planar boundary of a dielectric medium ($z \geq 0$) with $\varepsilon_{\mathrm{r}} = 9$.

Solution:

(a) $\widetilde{\mathbf{H}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 2 \times 10^{-2} e^{-j(8x+6z)}$.

Since $\mathbf{H}^{\mathrm{i}}$ is along $\hat{\mathbf{y}}$, which is perpendicular to the plane of incidence, the wave is TM polarized, or equivalently, its electric field vector is parallel polarized (parallel to the plane of incidence).

(b) From Eq. (8.65b), the argument of the exponential is

$$-jk_1(x \sin\theta_{\mathrm{i}} + z\cos\theta_{\mathrm{i}}) = -j(8x+6z).$$

Hence,

$$k_1 \sin\theta_{\mathrm{i}} = 8, \qquad k_1 \cos\theta_{\mathrm{i}} = 6,$$

from which we determine

$$\theta_{\mathrm{i}} = \tan^{-1}\left(\frac{8}{6}\right) = 53.13^\circ,$$
$$k_1 = \sqrt{6^2+8^2} = 10 \quad \text{(rad/m)}.$$

Also,

$$\omega = u_{\mathrm{p}} k = ck = 3\times 10^8 \times 10 = 3\times 10^9 \quad \text{(rad/s)}.$$

(c)

$$\eta_1 = \eta_0 = 377\ \Omega,$$
$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_2}}} = \frac{\eta_0}{3} = 125.67\ \Omega,$$
$$\theta_{\mathrm{t}} = \sin^{-1}\left[\frac{\sin\theta_{\mathrm{i}}}{\sqrt{\varepsilon_{\mathrm{r}_2}}}\right] = \sin^{-1}\left[\frac{\sin 53.13^\circ}{\sqrt{9}}\right] = 15.47^\circ,$$
$$\Gamma_{\parallel} = \frac{\eta_2\cos\theta_{\mathrm{t}} - \eta_1\cos\theta_{\mathrm{i}}}{\eta_2\cos\theta_{\mathrm{t}} + \eta_1\cos\theta_{\mathrm{i}}} = -0.30,$$
$$\tau_{\parallel} = (1+\Gamma_{\parallel})\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}} = 0.44.$$

In accordance with Eqs. (8.65a) to (8.65d), $E_0^{\mathrm{i}} = 2\times 10^{-2}\eta_1$ and

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = (\hat{\mathbf{x}}\cos\theta_{\mathrm{i}} - \hat{\mathbf{z}}\sin\theta_{\mathrm{i}})\, 2\times 10^{-2}\eta_1\, e^{-j(8x+6z)} = (\hat{\mathbf{x}}\, 4.52 - \hat{\mathbf{z}}\, 6.03)\, e^{-j(8x+6z)}.$$

$\widetilde{\mathbf{E}}^{\mathrm{r}}$ is similar to $\widetilde{\mathbf{E}}^{\mathrm{i}}$ except for reversal of z-components and multiplication of amplitude by $\Gamma_{\parallel}$. Hence, with $\Gamma_{\parallel} = -0.30$,

$$\begin{aligned}
\mathbf{E}^{\mathrm{r}} &= \mathfrak{Re}[\widetilde{\mathbf{E}}^{\mathrm{r}} e^{j\omega t}] = -(\hat{\mathbf{x}}\, 1.36 + \hat{\mathbf{z}}\, 1.81) \cos(3 \times 10^9 t - 8x + 6z) \text{ V/m}, \\
\mathbf{H}^{\mathrm{r}} &= \hat{\mathbf{y}}\, 2 \times 10^{-2} \Gamma_{\parallel} \cos(3 \times 10^9 t - 8x + 6z) \\
&= -\hat{\mathbf{y}}\, 0.6 \times 10^{-2} \cos(3 \times 10^9 t - 8x + 6z) \text{ A/m}.
\end{aligned}$$

(d) In medium 2,

$$\begin{aligned}
k_2 &= k_1 \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 10\sqrt{9} = 30 \text{ rad/m}, \\
\theta_{\mathrm{t}} &= \sin^{-1}\left[\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \sin\theta_{\mathrm{i}}\right] = \sin^{-1}\left[\frac{1}{3}\sin 53.13^\circ\right] = 15.47^\circ,
\end{aligned}$$

and the exponent of $\mathbf{E}^{\mathrm{t}}$ and $\mathbf{H}^{\mathrm{t}}$ is

$$-jk_2(x\sin\theta_{\mathrm{t}} + z\cos\theta_{\mathrm{t}}) = -j30(x\sin 15.47^\circ + z\cos 15.47^\circ) = -j(8x + 28.91z).$$

Hence,

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\mathrm{t}} &= (\hat{\mathbf{x}}\cos\theta_{\mathrm{t}} - \hat{\mathbf{z}}\sin\theta_{\mathrm{t}}) E_0^{\mathrm{i}} \tau_{\parallel} e^{-j(8x+28.91z)} \\
&= (\hat{\mathbf{x}}\, 0.96 - \hat{\mathbf{z}}\, 0.27)\, 2 \times 10^{-2} \times 377 \times 0.44\, e^{-j(8x+28.91z)} \\
&= (\hat{\mathbf{x}}\, 3.18 - \hat{\mathbf{z}}\, 0.90)\, e^{-j(8x+28.91z)}, \\
\widetilde{\mathbf{H}}^{\mathrm{t}} &= \hat{\mathbf{y}} \frac{E_0^{\mathrm{i}} \tau_{\parallel}}{\eta_2}\, e^{-j(8x+28.91z)} \\
&= \hat{\mathbf{y}}\, 2.64 \times 10^{-2}\, e^{-j(8x+28.91z)}, \\
\mathbf{E}^{\mathrm{t}} &= \mathfrak{Re}\{\widetilde{\mathbf{E}}^{\mathrm{t}} e^{j\omega t}\} \\
&= (\hat{\mathbf{x}}\, 3.18 - \hat{\mathbf{z}}\, 0.90) \cos(3 \times 10^9 t - 8x - 28.91z) \text{ V/m}, \\
\mathbf{H}^{\mathrm{t}} &= \hat{\mathbf{y}}\, 2.64 \times 10^{-2} \cos(3 \times 10^9 t - 8x - 28.91z) \text{ A/m}.
\end{aligned}$$

(e)

$$S_{\mathrm{av}}^{\mathrm{t}} = \frac{|E_0^{\mathrm{t}}|^2}{2\eta_2} = \frac{|H_0^{\mathrm{t}}|^2}{2}\eta_2 = \frac{(2.64 \times 10^{-2})^2}{2} \times 125.67 = 44 \text{ mW/m}^2.$$

Problem 8.28 Natural light is randomly polarized, which means that, on average, half the light energy is polarized along any given direction (in the plane orthogonal to the direction of propagation) and the other half of the energy is polarized along the direction orthogonal to the first polarization direction. Hence, when treating natural light incident upon a planar boundary, we can consider half of its energy to be in the form of parallel-polarized waves and the other half as perpendicularly polarized waves. Determine the fraction of the incident power reflected by the planar surface of a piece of glass with $n = 1.5$ when illuminated by natural light at 70°.

Solution: Assume the incident power is 1 W. Hence:

Incident power with parallel polarization $= 0.5$ W,
Incident power with perpendicular polarization $= 0.5$ W.

$\varepsilon_2/\varepsilon_1 = (n_2/n_1)^2 = n^2 = 1.5^2 = 2.25$. Equations (8.60) and (8.68) give

$$\Gamma_\perp = \frac{\cos 70^\circ - \sqrt{2.25 - \sin^2 70^\circ}}{\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}} = -0.55,$$

$$\Gamma_\parallel = \frac{-2.25\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}}{2.25\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}} = 0.21.$$

Reflected power with parallel polarization $= 0.5\,(\Gamma_\parallel)^2$
$= 0.5\,(0.21)^2 = 22$ mW,
Reflected power with perpendicular polarization $= 0.5\,(\Gamma_\perp)^2$
$= 0.5\,(0.55)^2 = 151.3$ mW.

Total reflected power $= 22 + 151.3 = 173.3$ mW, or 17.33%..

Problem 8.29 A parallel polarized plane wave is incident from air onto a dielectric medium with $\varepsilon_r = 9$ at the Brewster angle. What is the refraction angle?

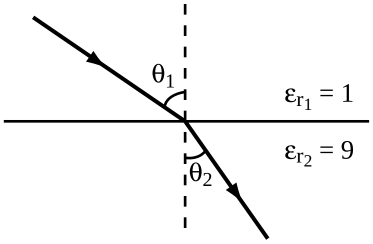


Figure P8.29: Geometry of Problem 8.29.

Solution: For nonmagnetic materials, Eq. (8.72) gives

$$\theta_1 = \theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon 1}} = \tan^{-1} 3 = 71.57^\circ.$$

But

$$\sin\theta_2 = \frac{\sin\theta_1}{\sqrt{\varepsilon_{r_2}}} = \frac{\sin\theta_1}{3} = \frac{\sin 71.57^\circ}{3} = 0.32,$$

or $\theta_2 = 18.44^\circ$.

Problem 8.30 A perpendicularly polarized wave in air is obliquely incident upon a planar glass-air interface at an incidence angle of 30°. The wave frequency is 600 THz (1 THz $= 10^{12}$ Hz), which corresponds to green light, and the index of refraction of the glass is 1.6. If the electric field amplitude of the incident wave is 50 V/m, determine

(a) the reflection and transmission coefficients, and

(b) the instantaneous expressions for **E** and **H** in the glass medium.

Solution:

(a) For nonmagnetic materials, $(\varepsilon_2/\varepsilon_1) = (n_2/n_1)^2$. Using this relation in Eq. (8.60) gives

$$\Gamma_\perp = \frac{\cos\theta_i - \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(n_2/n_1)^2 - \sin^2\theta_i}} = \frac{\cos 30^\circ - \sqrt{(1.6)^2 - \sin^2 30^\circ}}{\cos 30^\circ + \sqrt{(1.6)^2 - \sin^2 30^\circ}} = -0.27,$$

$$\tau_\perp = 1 + \Gamma_\perp = 1 - 0.27 = 0.73.$$

(b) In the glass medium,

$$\sin\theta_t = \frac{\sin\theta_i}{n_2} = \frac{\sin 30^\circ}{1.6} = 0.31,$$

or $\theta_t = 18.21^\circ$.

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \frac{\eta_0}{n_2} = \frac{120\pi}{1.6} = 75\pi = 235.62 \quad (\Omega),$$

$$k_2 = \frac{\omega}{u_p} = \frac{2\pi f}{c/n} = \frac{2\pi f n}{c} = \frac{2\pi \times 600 \times 10^{12} \times 1.6}{3 \times 10^8} = 6.4\pi \times 10^6 \text{ rad/m},$$

$$E_0^t = \tau_\perp E_0^i = 0.73 \times 50 = 36.5 \text{ V/m}.$$

From Eqs. (8.49c) and (8.49d),

$$\widetilde{\mathbf{E}}_\perp^t = \hat{\mathbf{y}} E_0^t e^{-jk_2(x\sin\theta_t + z\cos\theta_t)},$$

$$\widetilde{\mathbf{H}}_\perp^t = (-\hat{\mathbf{x}}\cos\theta_t + \hat{\mathbf{z}}\sin\theta_t)\frac{E_0^t}{\eta_2} e^{-jk_2(x\sin\theta_t + z\cos\theta_t)},$$

and the corresponding instantaneous expressions are:

$$\mathbf{E}^{\mathrm{t}}_{\perp}(x,z,t) = \hat{\mathbf{y}}36.5\cos(\omega t - k_2 x\sin\theta_{\mathrm{t}} - k_2 z\cos\theta_{\mathrm{t}}) \quad \text{(V/m)},$$
$$\mathbf{H}^{\mathrm{t}}_{\perp}(x,z,t) = (-\hat{\mathbf{x}}\cos\theta_{\mathrm{t}} - \hat{\mathbf{z}}\cos\theta_{\mathrm{t}})0.16\cos(\omega t - k_2 x\sin\theta_{\mathrm{t}} - k_2 z\cos\theta_{\mathrm{t}}) \quad \text{(A/m)},$$

with $\omega = 2\pi\times 10^{15}$ rad/s and $k_2 = 6.4\pi\times 10^6$ rad/m.

Problem 8.31 Show that the reflection coefficient $\Gamma_{\perp}$ can be written in the form

$$\Gamma_{\perp} = \frac{\sin(\theta_{\mathrm{t}} - \theta_{\mathrm{i}})}{\sin(\theta_{\mathrm{t}} + \theta_{\mathrm{i}})}\,.$$

Solution: From Eq. (8.58a),

$$\Gamma_{\perp} = \frac{\eta_2\cos\theta_{\mathrm{i}} - \eta_1\cos\theta_{\mathrm{t}}}{\eta_2\cos\theta_{\mathrm{i}} + \eta_1\cos\theta_{\mathrm{t}}} = \frac{(\eta_2/\eta_1)\cos\theta_{\mathrm{i}} - \cos\theta_{\mathrm{t}}}{(\eta_2/\eta_1)\cos\theta_{\mathrm{i}} + \cos\theta_{\mathrm{t}}}\,.$$

Using Snell's law for refraction given by Eq. (8.31), we have

$$\frac{\eta_2}{\eta_1} = \frac{\sin\theta_{\mathrm{t}}}{\sin\theta_{\mathrm{i}}}\,,$$

we have

$$\Gamma_{\perp} = \frac{\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{i}} - \cos\theta_{\mathrm{t}}\sin\theta_{\mathrm{i}}}{\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{i}} + \cos\theta_{\mathrm{t}}\sin\theta_{\mathrm{i}}} = \frac{\sin(\theta_{\mathrm{t}} - \theta_{\mathrm{i}})}{\sin(\theta_{\mathrm{t}} + \theta_{\mathrm{i}})}\,.$$

Problem 8.32 Show that for nonmagnetic media, the reflection coefficient $\Gamma_{\parallel}$ can be written in the form

$$\Gamma_{\parallel} = \frac{\tan(\theta_{\mathrm{t}} - \theta_{\mathrm{i}})}{\tan(\theta_{\mathrm{t}} + \theta_{\mathrm{i}})}\,.$$

Solution: From Eq. (8.66a), $\Gamma_{\parallel}$ is given by

$$\Gamma_{\parallel} = \frac{\eta_2\cos\theta_{\mathrm{t}} - \eta_1\cos\theta_{\mathrm{i}}}{\eta_2\cos\theta_{\mathrm{t}} + \eta_1\cos\theta_{\mathrm{i}}} = \frac{(\eta_2/\eta_1)\cos\theta_{\mathrm{t}} - \cos\theta_{\mathrm{i}}}{(\eta_2/\eta_1)\cos\theta_{\mathrm{t}} + \cos\theta_{\mathrm{i}}}\,.$$

For nonmagnetic media, $\mu_1 = \mu_2 = \mu_0$ and

$$\frac{\eta_2}{\eta_1} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \frac{n_1}{n_2}\,.$$

Snell's law of refraction is

$$\frac{\sin\theta_{\rm t}}{\sin\theta_{\rm i}} = \frac{n_1}{n_2}.$$

Hence,

$$\Gamma_\parallel = \frac{\dfrac{\sin\theta_{\rm t}}{\sin\theta_{\rm i}}\cos\theta_{\rm t} - \cos\theta_{\rm i}}{\dfrac{\sin\theta_{\rm t}}{\sin\theta_{\rm i}}\cos\theta_{\rm t} + \cos\theta_{\rm i}} = \frac{\sin\theta_{\rm t}\cos\theta_{\rm t} - \sin\theta_{\rm i}\cos\theta_{\rm i}}{\sin\theta_{\rm t}\cos\theta_{\rm t} + \sin\theta_{\rm i}\cos\theta_{\rm i}}.$$

To show that the expression for $\Gamma_\parallel$ is the same as

$$\Gamma_\parallel = \frac{\tan(\theta_{\rm t} - \theta_{\rm i})}{\tan(\theta_{\rm t} + \theta_{\rm i})},$$

we shall proceed with the latter and show that it is equal to the former.

$$\frac{\tan(\theta_{\rm t} - \theta_{\rm i})}{\tan(\theta_{\rm t} + \theta_{\rm i})} = \frac{\sin(\theta_{\rm t} - \theta_{\rm i})\cos(\theta_{\rm t} + \theta_{\rm i})}{\cos(\theta_{\rm t} - \theta_{\rm i})\sin(\theta_{\rm t} + \theta_{\rm i})}.$$

Using the identities (from Appendix C):

$$2\sin x\cos y = \sin(x+y) + \sin(x-y),$$

and if we let $x = \theta_{\rm t} - \theta_{\rm i}$ and $y = \theta_{\rm t} + \theta_{\rm i}$ in the numerator, while letting $x = \theta_{\rm t} + \theta_{\rm i}$ and $y = \theta_{\rm t} - \theta_{\rm i}$ in the denominator, then

$$\frac{\tan(\theta_{\rm t} - \theta_{\rm i})}{\tan(\theta_{\rm t} + \theta_{\rm i})} = \frac{\sin(2\theta_{\rm t}) + \sin(-2\theta_{\rm i})}{\sin(2\theta_{\rm t}) + \sin(2\theta_{\rm i})}.$$

But $\sin 2\theta = 2\sin\theta\cos\theta$, and $\sin(-\theta) = -\sin\theta$, hence,

$$\frac{\tan(\theta_{\rm t} - \theta_{\rm i})}{\tan(\theta_{\rm t} + \theta_{\rm i})} = \frac{\sin\theta_{\rm t}\cos\theta_{\rm t} - \sin\theta_{\rm i}\cos\theta_{\rm i}}{\sin\theta_{\rm t}\cos\theta_{\rm t} + \sin\theta_{\rm i}\cos\theta_{\rm i}},$$

which is the intended result.

Problem 8.33 A parallel polarized beam of light with an electric field amplitude of 10 (V/m) is incident in air on polystyrene with $\mu_{\rm r} = 1$ and $\varepsilon_{\rm r} = 2.6$. If the incidence angle at the air–polystyrene planar boundary is 50°, determine

(a) the reflectivity and transmissivity, and

(b) the power carried by the incident, reflected, and transmitted beams if the spot on the boundary illuminated by the incident beam is 1 m^2 in area.

Solution:

(a) From Eq. (8.68),

$$\begin{aligned}
\Gamma_{\parallel} &= \frac{-(\varepsilon_2/\varepsilon_1)\cos\theta_{\rm i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_{\rm i}}}{(\varepsilon_2/\varepsilon_1)\cos\theta_{\rm i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_{\rm i}}} \\
&= \frac{-2.6\cos 50^\circ + \sqrt{2.6 - \sin^2 50^\circ}}{2.6\cos 50^\circ + \sqrt{2.6 - \sin^2 50^\circ}} = -0.08, \\
R_{\parallel} &= |\Gamma_{\parallel}|^2 = (0.08)^2 = 6.4\times 10^{-3}, \\
T_{\parallel} &= 1 - R_{\parallel} = 0.9936.
\end{aligned}$$

(b)

$$\begin{aligned}
P^{\rm i}_{\parallel} &= \frac{|E^{\rm i}_{\parallel 0}|^2}{2\eta_1} A\cos\theta_{\rm i} = \frac{(10)^2}{2\times 120\pi} \times \cos 50^\circ = 85 \text{ mW}, \\
P^{\rm r}_{\parallel} &= R_{\parallel} P^{\rm i}_{\parallel} = (6.4\times 10^{-3}) \times 0.085 = 0.55 \text{ mW}, \\
P^{\rm t}_{\parallel} &= T_{\parallel} P^{\rm i}_{\parallel} = 0.9936 \times 0.085 = 84.45 \text{ mW}.
\end{aligned}$$

Sections 8-6 to 8-11

Problem 8.34 Derive Eq. (8.89b).

Solution:

We start with Eqs. (8.88a and e),

$$\frac{\partial \widetilde{e}_z}{\partial y} + j\beta\widetilde{e}_y = -j\omega\mu\widetilde{h}_x,$$

$$-j\beta\widetilde{h}_x - \frac{\partial \widetilde{h}_z}{\partial x} = j\omega\varepsilon\widetilde{e}_y.$$

To eliminate $\widetilde{h}_x$, we multiply the top equation by β and the bottom equation by $\omega\mu$, and then we add them together. The result is:

$$\beta\frac{\partial \widetilde{e}_z}{\partial y} + j\beta^2\widetilde{e}_y - \omega\mu\frac{\partial \widetilde{h}_z}{\partial x} = j\omega^2\mu\varepsilon\widetilde{e}_y.$$

Multiplying all terms by $e^{-j\beta z}$ to convert $\widetilde{e}_y$ to $\widetilde{E}_y$ (and similarly for the other field components), and then solving for $\widetilde{E}_y$ leads to

$$\begin{aligned}\widetilde{E}_y &= \frac{1}{j(\beta^2 - \omega^2\mu\varepsilon)}\left(-\beta\,\frac{\partial\widetilde{E}_z}{\partial y} + \omega\mu\,\frac{\partial\widetilde{H}_z}{\partial x}\right) \\ &= \frac{j}{k_c^2}\left(-\beta\,\frac{\partial\widetilde{E}_z}{\partial y} + \omega\mu\,\frac{\partial\widetilde{H}_z}{\partial x}\right),\end{aligned}$$

where we used the relation

$$k_c^2 = \omega^2\mu\varepsilon - \beta^2.$$

Problem 8.35 A hollow rectangular waveguide is to be used to transmit signals at a carrier frequency of 6 GHz. Choose its dimensions so that the cutoff frequency of the dominant TE mode is lower than the carrier by 25% and that of the next mode is at least 25% higher than the carrier.

Solution:

For $m = 1$ and $n = 0$ (TE$_{10}$ mode) and $u_{p_0} = c$ (hollow guide), Eq. (8.106) reduces to

$$f_{10} = \frac{c}{2a}\,.$$

Denote the carrier frequency as $f_0 = 6$ GHz. Setting

$$f_{10} = 0.75 f_0 = 0.75 \times 6 \text{ GHz} = 4.5 \text{ GHz},$$

we have

$$a = \frac{c}{2f_{10}} = \frac{3\times 10^8}{2\times 4.5\times 10^9} = 3.33 \text{ cm}.$$

If b is chosen such that $a > b > \frac{a}{2}$, the second mode will be TE$_{01}$, followed by TE$_{20}$ at $f_{20} = 9$ GHz. For TE$_{01}$,

$$f_{01} = \frac{c}{2b}\,.$$

Setting $f_{01} = 1.25 f_0 = 7.5$ GHz, we get

$$b = \frac{c}{2f_{01}} = \frac{3\times 10^8}{2\times 7.5\times 10^9} = 2 \text{ cm}.$$

Problem 8.36 A TE wave propagating in a dielectric-filled waveguide of unknown permittivity has dimensions $a = 5$ cm and $b = 3$ cm. If the x-component of its electric field is given by

$$\begin{aligned} E_x &= -36\cos(40\pi x)\sin(100\pi y) \\ &\quad \cdot \sin(2.4\pi \times 10^{10} t - 52.9\pi z), \quad \text{(V/m)} \end{aligned}$$

determine:

(a) the mode number,
(b) ε_r of the material in the guide,
(c) the cutoff frequency, and
(d) the expression for H_y.

Solution:

(a) Comparison of the given expression with Eq. (8.110a) reveals that

$$\frac{m\pi}{a} = 40\pi, \qquad \text{hence } m = 2$$
$$\frac{n\pi}{b} = 100\pi, \qquad \text{hence } n = 3.$$

Mode is TE_{23}.

(b) From $\sin(\omega t - \beta z)$, we deduce that

$$\omega = 2.4\pi \times 10^{10} \text{ rad/s}, \qquad \beta = 52.9\pi \text{ rad/m}.$$

Using Eq. (8.105) to solve for ε_r, we have

$$\begin{aligned} \varepsilon_\mathrm{r} &= \frac{c^2}{\omega^2}\left[\beta^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \\ &= 2.25. \end{aligned}$$

(c)

$$u_{\mathrm{p}_0} = \frac{c}{\sqrt{\varepsilon_\mathrm{r}}} = \frac{3 \times 10^8}{\sqrt{2.25}} = 2 \times 10^8 \text{ m/s}.$$

$$\begin{aligned} f_{23} &= \frac{u_{\mathrm{p}_0}}{2}\sqrt{\left(\frac{2}{a}\right)^2 + \left(\frac{3}{b}\right)^2} \\ &= 10.77 \text{ GHz}. \end{aligned}$$

(d)

$$Z_{\text{TE}} = \frac{E_x}{H_y} = \eta \bigg/ \sqrt{1-(f_{23}/f)^2}$$

$$= \frac{377}{\sqrt{\varepsilon_r}} \Bigg/ \sqrt{1-\left(\frac{10.77}{12}\right)^2} = 569.9\ \Omega.$$

Hence,

$$\begin{aligned} H_y &= \frac{E_x}{Z_{\text{TE}}} \\ &= -0.063\cos(40\pi x)\sin(100\pi y)\sin(2.4\pi\times 10^{10}t - 52.9\pi z) \quad \text{(A/m)}. \end{aligned}$$

Problem 8.37 A waveguide filled with a material whose $\varepsilon_r = 2.25$ has dimensions $a = 2$ cm and $b = 1.4$ cm. If the guide is to transmit 10.5-GHz signals, what possible modes can be used for the transmission?

Solution:
Application of Eq. (8.106) with $u_{p_0} = c/\sqrt{\varepsilon_r} = 3\times 10^8/\sqrt{2.25} = 2\times 10^8$ m/s, gives:

$$\begin{aligned} f_{10} &= 5 \text{ GHz (TE only)} \\ f_{01} &= 7.14 \text{ GHz (TE only)} \\ f_{11} &= 8.72 \text{ GHz (TE or TM)} \\ f_{20} &= 10 \text{ GHz (TE only)} \\ f_{21} &= 12.28 \text{ GHz (TE or TM)} \\ f_{12} &= 15.1 \text{ GHz (TE or TM)}. \end{aligned}$$

Hence, any one of the first four modes can be used to transmit 10.5-GHz signals.

Problem 8.38 For a rectangular waveguide operating in the TE_{10} mode, obtain expressions for the surface charge density $\widetilde{\rho}_s$ and surface current density $\widetilde{\mathbf{J}}_s$ on each of the four walls of the guide.

Solution:

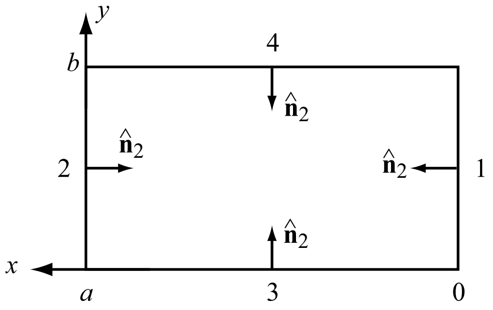
For TE$_{10}$, the expressions for $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ are given by Eq. (8.110) with $m = 1$ and $n = 0$,

$$\begin{aligned}
\widetilde{E}_x &= 0,\\
\widetilde{E}_y &= -j\frac{\omega\mu\pi H_0}{k_c^2 a}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z},\\
\widetilde{E}_z &= 0,\\
\widetilde{H}_x &= j\frac{\beta\pi H_0}{k_c^2 a}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z},\\
\widetilde{H}_y &= 0,\\
\widetilde{H}_z &= H_0\cos\left(\frac{\pi x}{a}\right)e^{-j\beta z}.
\end{aligned}$$

The applicable boundary conditions are given in Table 6-2. At the boundary between a dielectric (medium 1) and a conductor (medium 2),

$$\begin{aligned}
\widetilde{\rho}_s &= \hat{\mathbf{n}}_2\cdot\widetilde{\mathbf{D}}_1 = \varepsilon_1\,\hat{\mathbf{n}}_2\cdot\widetilde{\mathbf{E}}_1,\\
\widetilde{\mathbf{J}}_s &= \hat{\mathbf{n}}_2\times\widetilde{\mathbf{H}}_1,
\end{aligned}$$

where $\widetilde{\mathbf{E}}_1$ and $\widetilde{\mathbf{H}}_1$ are the fields inside the guide, ε_1 is the permittivity of the material filling the guide, and $\hat{\mathbf{n}}_2$ is the normal to the guide wall, pointing away from the wall (inwardly). In view of the coordinate system defined for the guide, $\hat{\mathbf{n}}_2 = \hat{\mathbf{x}}$ for side wall at $x = 0$, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{x}}$ for wall at $x = a$, etc.



(a) At side wall 1 at $x = 0$, $\hat{\mathbf{n}}_2 = \hat{\mathbf{x}}$. Hence,

$$\begin{aligned}
\rho_s &= \varepsilon_1\,\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}E_y|_{x=0} = 0\\
\mathbf{J}_s &= \hat{\mathbf{x}}\times(\hat{\mathbf{x}}\widetilde{H}_x + \hat{\mathbf{z}}\widetilde{H}_z)|_{x=0}\\
&= -\hat{\mathbf{y}}\widetilde{H}_z|_{x=0}\\
&= -\hat{\mathbf{y}}H_0e^{-j\beta z}.
\end{aligned}$$

(b) At side wall 2 at $x = a$, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{x}}$. Hence,

$$\begin{aligned}
\rho_{\text{s}} &= 0 \\
\mathbf{J}_{\text{s}} &= \hat{\mathbf{y}} H_0 e^{-j\beta z}.
\end{aligned}$$

(c) At bottom surface at $y = 0$, $\hat{\mathbf{n}}_2 = \hat{\mathbf{y}}$. Hence,

$$\begin{aligned}
\rho_{\text{s}} &= \varepsilon_1 \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} E_y|_{y=0} \\
&= -j \frac{\omega\varepsilon\mu\pi H_0}{k_{\text{c}}^2 a} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z} \\
\widetilde{\mathbf{J}}_{\text{s}} &= \hat{\mathbf{y}} \times (\hat{\mathbf{x}} \widetilde{H}_x + \hat{\mathbf{z}} \widetilde{H}_z) \\
&= H_0 \left[\hat{\mathbf{x}} \cos\left(\frac{\pi x}{a}\right) - \hat{\mathbf{z}}\, j \frac{\beta\pi}{k_{\text{c}}^2 a} \sin\left(\frac{\pi x}{a}\right) \right] e^{-j\beta z}.
\end{aligned}$$

(d) At top surface at $y - b$, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{y}}$. Hence,

$$\begin{aligned}
\widetilde{\rho}_{\text{s}} &= j \frac{\omega\varepsilon\mu\pi H_0}{k_{\text{c}}^2 a} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z} \\
\widetilde{\mathbf{J}}_{\text{s}} &= H_0 \left[-\hat{\mathbf{x}} \cos\left(\frac{\pi x}{a}\right) + \hat{\mathbf{z}}\, j \frac{\beta\pi}{k_{\text{c}}^2 a} \sin\left(\frac{\pi x}{a}\right) \right] e^{-j\beta z}.
\end{aligned}$$

Problem 8.39 A waveguide, with dimensions $a = 1$ cm and $b = 0.7$ cm, is to be used at 20 GHz. Determine the wave impedance for the dominant mode when

(a) the guide is empty, and

(b) the guide is filled with polyethylene (whose $\varepsilon_{\text{r}} = 2.25$).

Solution:

For the TE_{10} mode,

$$f_{10} = \frac{u_{\text{p}_0}}{2a} = \frac{c}{2a\sqrt{\varepsilon_{\text{r}}}}\,.$$

When empty,

$$f_{10} = \frac{3 \times 10^8}{2 \times 10^{-2}} = 15 \text{ GHz}.$$

When filled with polyethylene, $f_{10} = 10$ GHz.

According to Eq. (8.111),

$$Z_{\text{TE}} = \frac{\eta}{\sqrt{1 - (f_{10}/f)^2}} = \frac{\eta_0}{\sqrt{\varepsilon_{\text{r}}}\sqrt{1 - (f_{10}/f)^2}}\,.$$

When empty,

$$Z_{\text{TE}} = \frac{377}{\sqrt{1-(15/20)^2}} = 570\ \Omega.$$

When filled,

$$Z_{\text{TE}} = \frac{377}{\sqrt{2.25}\sqrt{1-(10/20)^2}} = 290\ \Omega.$$

Problem 8.40 A narrow rectangular pulse superimposed on a carrier with a frequency of 9.5 GHz was used to excite all possible modes in a hollow guide with $a = 3$ cm and $b = 2.0$ cm. If the guide is 100 m in length, how long will it take each of the excited modes to arrive at the receiving end?

Solution:

With $a = 3$ cm, $b = 2$ cm, and $u_{\text{p}_0} = c = 3\times10^8$ m/s, application of Eq. (8.106) leads to:

$$\begin{aligned} f_{10} &= 5 \text{ GHz} \\ f_{01} &= 7.5 \text{ GHz} \\ f_{11} &= 9.01 \text{ GHz} \\ f_{20} &= 10 \text{ GHz} \end{aligned}$$

Hence, the pulse with a 9.5-GHz carrier can excite the top three modes. Their group velocities can be calculated with the help of Eq. (8.114),

$$u_{\text{g}} = c\sqrt{1-(f_{mn}/f)^2},$$

which gives:

$$u_{\text{g}} = \begin{cases} 0.85c = 2.55\times10^8 \text{ m/s}, & \text{for TE}_{10} \\ 0.61c = 1.84\times10^8 \text{ m/s}, & \text{for TE}_{01} \\ 0.32c = 0.95\times10^8 \text{ m/s}, & \text{for TE}_{11} \text{ and TM}_{11} \end{cases}$$

Travel time associated with these modes is:

$$T = \frac{d}{u_{\text{g}}} = \frac{100}{u_{\text{g}}} = \begin{cases} 0.39\ \mu\text{s}, & \text{for TE}_{10} \\ 0.54\ \mu\text{s}, & \text{for TE}_{01} \\ 1.05\ \mu\text{s}, & \text{for TE}_{11} \text{ and TM}_{11}. \end{cases}$$

Problem 8.41 If the zigzag angle θ' is 42° for the TE_{10} mode, what would it be for the TE_{20} mode?

Solution:

For TE_{10}, the derivation that started with Eq. (8.116) led to

$$\theta'_{10} = \tan^{-1}\left(\frac{\pi}{\beta a}\right), \qquad \mathrm{TE}_{10} \text{ mode.}$$

Had the derivation been for $n = 2$ (instead of $n = 1$), the x-dependence would have involved a phase factor $(2\pi x/a)$ (instead of $(\pi x/a)$). The sequence of steps would have led to

$$\theta'_{20} = \tan^{-1}\left(\frac{2\pi}{\beta a}\right), \qquad \mathrm{TE}_{20} \text{ mode.}$$

Given that $\theta'_{10} = 42^\circ$, it follows that

$$\frac{\pi}{\beta a} = \tan 42^\circ = 0.90$$

Hence,

$$\theta'_{20} = \tan^{-1}(2 \times 0.9) = 60.9^\circ.$$

Problem 8.42 Measurement of the TE_{101} frequency response of an air-filled cubic cavity revealed that its Q is 4802. If its volume is 64 mm^3, what material are its sides made of?

Solution:

According to Eq. (8.121), the TE_{101} resonant frequency of a cubic cavity is given by

$$f_{101} = \frac{3 \times 10^8}{\sqrt{2}\, a} = \frac{3 \times 10^8}{\sqrt{2} \times 4 \times 10^{-3}} = 53.0 \text{ GHz.}$$

Its Q is given by

$$Q = \frac{a}{3\delta_s} = 4802,$$

which gives $\delta_s = 2.78 \times 10^{-7}$ m. Applying

$$\delta_s = \frac{1}{\sqrt{\pi f_{101} \mu_0 \sigma_c}},$$

and solving for σ_c leads to

$$\sigma_c \simeq 6.2 \times 10^7 \text{ S/m.}$$

According to Appendix B, the material is silver.

Problem 8.43 A hollow cavity made of aluminum has dimensions $a = 4$ cm and $d = 3$ cm. Calculate Q of the TE_{101} mode for

(a) $b = 2$ cm, and

(b) $b = 3$ cm.

Solution:

For the TE_{101} mode, f_{101} is independent of b,

$$\begin{aligned} f_{101} &= \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2} \\ &= \frac{3\times 10^8}{2}\sqrt{\left(\frac{1}{4\times 10^{-2}}\right)^2 + \left(\frac{1}{3\times 10^{-2}}\right)^2} \\ &= 6.25 \text{ GHz.} \end{aligned}$$

For aluminum with $\sigma_c = 3.5\times 10^7$ S/m (Appendix B),

$$\delta_s = \frac{1}{\sqrt{\pi f_{101}\mu_0\sigma_c}} = 1.08\times 10^{-6} \text{ m.}$$

(a) For $a = 4$ cm, $b = 2$ cm and $d = 3$ cm,

$$\begin{aligned} Q &= \frac{1}{\delta_s}\,\frac{abd(a^2+d^2)}{[a^3(d+2b)+d^3(a+2b)]} \\ &= 8367. \end{aligned}$$

(b) For $a = 4$ cm, $b = 3$ cm, and $d = 3$ cm,

$$Q = 9850.$$

Problem 8.44 A 50-MHz right-hand circularly polarized plane wave with an electric field modulus of 30 V/m is normally incident in air upon a dielectric medium with $\varepsilon_r = 9$ and occupying the region defined by $z \geq 0$.

(a) Write an expression for the electric field phasor of the incident wave, given that the field is a positive maximum at $z = 0$ and $t = 0$.

(b) Calculate the reflection and transmission coefficients.

(c) Write expressions for the electric field phasors of the reflected wave, the transmitted wave, and the total field in the region $z \leq 0$.

(d) Determine the percentages of the incident average power reflected by the boundary and transmitted into the second medium.

Solution:

(a)

$$\begin{aligned} k_1 &= \frac{\omega}{c} = \frac{2\pi \times 50 \times 10^6}{3 \times 10^8} = \frac{\pi}{3} \text{ rad/m}, \\ k_2 &= \frac{\omega}{c}\sqrt{\varepsilon_{\text{r}_2}} = \frac{\pi}{3}\sqrt{9} = \pi \text{ rad/m}. \end{aligned}$$

From (7.57), RHC wave traveling in $+z$ direction:

$$\begin{aligned} \widetilde{\mathbf{E}}^{\text{i}} &= a_0(\hat{\mathbf{x}} + \hat{\mathbf{y}} e^{-j\pi/2})e^{-jk_1z} = a_0(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{-jk_1z} \\ \mathbf{E}^{\text{i}}(z,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}^{\text{i}} e^{j\omega t}\right] \\ &= \mathfrak{Re}\left[a_0(\hat{\mathbf{x}} e^{j(\omega t - k_1 z)} + \hat{\mathbf{y}} e^{j(\omega t - k_1 z - \pi/2)})\right] \\ &= \hat{\mathbf{x}} a_0 \cos(\omega t - k_1 z) + \hat{\mathbf{y}} a_0 \cos(\omega t - k_1 z - \pi/2) \\ &= \hat{\mathbf{x}} a_0 \cos(\omega t - k_1 z) + \hat{\mathbf{y}} a_0 \sin(\omega t - k_1 z) \\ |\mathbf{E}^{\text{i}}| &= \left[a_0^2 \cos^2(\omega t - k_1 z) + a_0^2 \sin^2(\omega t - k_1 z)\right]^{1/2} = a_0 = 30 \text{ V/m}. \end{aligned}$$

Hence,

$$\widetilde{\mathbf{E}}^{\text{i}} = 30(x_0 - jy_0)e^{-j\pi z/3} \quad \text{(V/m)}.$$

(b)

$$\eta_1 = \eta_0 = 120\pi \quad (\Omega), \qquad \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{\text{r}_2}}} = \frac{120\pi}{\sqrt{9}} = 40\pi \quad (\Omega).$$

$$\begin{aligned} \Gamma &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{40\pi - 120\pi}{40\pi + 120\pi} = -0.5 \\ \tau &= 1 + \Gamma = 1 - 0.5 = 0.5. \end{aligned}$$

(c)

$$\begin{aligned}
\widetilde{\mathbf{E}}^{\rm r} &= \Gamma a_0(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{jk_1z} \\
&= -0.5 \times 30(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{jk_1z} \\
&= -15(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{j\pi z/3} \quad \text{(V/m)}. \\
\widetilde{\mathbf{E}}^{\rm t} &= \tau a_0(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{-jk_2z} \\
&= 15(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{-j\pi z} \quad \text{(V/m)}. \\
\widetilde{\mathbf{E}}_1 &= \widetilde{\mathbf{E}}^{\rm i} + \widetilde{\mathbf{E}}^{\rm r} \\
&= 30(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{-j\pi z/3} - 15(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{j\pi z/3} \\
&= 15(\hat{\mathbf{x}} - j\hat{\mathbf{y}})[2e^{-j\pi z/3} - e^{j\pi z/3}] \quad \text{(V/m)}.
\end{aligned}$$

(d)

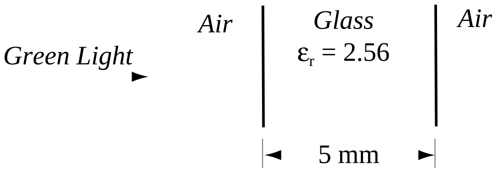
$$\begin{aligned}
\%\text{ of reflected power} &= 100 \times |\Gamma|^2 = 100 \times (0.5)^2 = 25\% \\
\%\text{ of transmitted power} &= 100|\tau|^2 \frac{\eta_1}{\eta_2} = 100 \times (0.5)^2 \times \frac{120\pi}{40\pi} = 75\%.
\end{aligned}$$

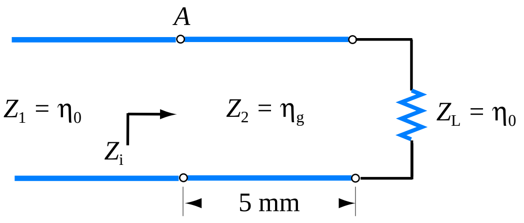
Problem 8.45 Consider a flat 5-mm-thick slab of glass with $\varepsilon_{\rm r} = 2.56$.

(a) If a beam of green light ($\lambda_0 = 0.52$ μm) is normally incident upon one of the sides of the slab, what percentage of the incident power is reflected back by the glass?

(b) To eliminate reflections, it is desired to add a thin layer of antireflection coating material on each side of the glass. If you are at liberty to specify the thickness of the antireflection material as well as its relative permittivity, what would these specifications be?

Solution:





(a) Representing the wave propagation process by an equivalent transmission line model, the input impedance at the left-hand side of the air-glass interface is (from 2.63):

$$Z_\mathrm{i} = Z_0\left(\frac{Z_\mathrm{L} + jZ_0\tan\beta l}{Z_0 + jZ_\mathrm{L}\tan\beta l}\right)$$

For the glass,

$$Z_0 = \eta_\mathrm{g} = \frac{\eta_0}{\sqrt{\varepsilon_\mathrm{r}}} = \frac{\eta_0}{\sqrt{2.56}} = \frac{\eta_0}{1.6}$$

$$Z_\mathrm{L} = \eta_0$$

$$\beta l = \frac{2\pi}{\lambda}l = \frac{2\pi}{\lambda_0}\sqrt{\varepsilon_\mathrm{r}}\,l = \frac{2\pi}{0.52\times10^{-6}}\times\sqrt{2.56}\times5\times10^{-3} = 30769.23\pi.$$

Subtracting the maximum possible multiples of 2π, namely 30768π, leaves a remainder of

$$\beta l = 1.23\pi \text{ rad}.$$

Hence,

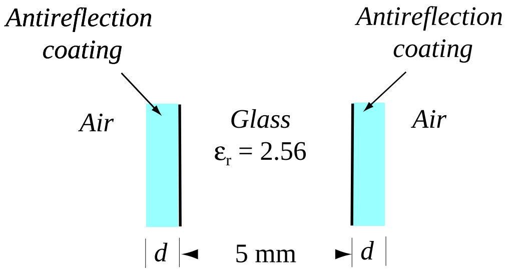
$$\begin{aligned}
Z_\mathrm{i} &= \frac{\eta_0}{1.6}\left(\frac{\eta_0 + j(\eta_0/1.6)\tan 1.23\pi}{(\eta_0/1.6) + j\eta_0\tan 1.23\pi}\right)\\
&= \left(\frac{1.6 + j\tan 1.23\pi}{1 + j1.6\tan 1.23\pi}\right)\frac{120\pi}{1.6}\\
&= \left(\frac{1.6 + j0.882}{1 + j1.41}\right)\frac{120\pi}{1.6} = 249\angle{-25.8^\circ} = (224.2 - j108.4)\ \Omega.
\end{aligned}$$

With Z_i now representing the input impedance of the glass, the reflection coefficient at point A is:

$$\begin{aligned}\Gamma &= \frac{Z_\mathrm{i}-\eta_0}{Z_\mathrm{i}+\eta_0} \\ &= \frac{224.2-j108.4-120\pi}{224.2-j108.4+120\pi} = \frac{187.34\angle{-144.6^\circ}}{610.89\angle{-10.2^\circ}} = 0.3067\angle{-154.8^\circ}\,.\end{aligned}$$

% of reflected power $= |\Gamma|^2 \times 100 = 9.4\%$.

(b) To avoid reflections, we can add a quarter-wave transformer on each side of the glass.



This requires that d be:

$$d = \frac{\lambda}{4} + 2n\lambda, \qquad n = 0,1,2,\ldots$$

where λ is the wavelength in that material; i.e., $\lambda = \lambda_0/\sqrt{\varepsilon_\mathrm{rc}}$, where ε_rc is the relative permittivity of the coating material. It is also required that η_c of the coating material be:

$$\eta_\mathrm{c}^2 = \eta_0 \eta_\mathrm{g}.$$

Thus

$$\frac{\eta_0^2}{\varepsilon_\mathrm{rc}} = \eta_0 \frac{\eta_0}{\sqrt{\varepsilon_\mathrm{r}}}\,,$$

or

$$\varepsilon_\mathrm{rc} = \sqrt{\varepsilon_\mathrm{r}} = \sqrt{2.56} = 1.6.$$

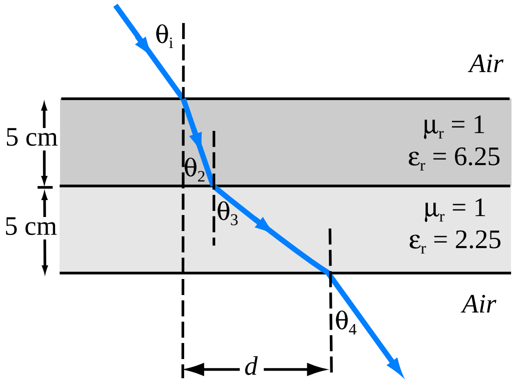
Hence,

$$\begin{aligned}\lambda &= \frac{\lambda_0}{\sqrt{\varepsilon_\mathrm{rc}}} = \frac{0.52\ \mu\mathrm{m}}{\sqrt{1.6}} = 0.411\ \mu\mathrm{m}, \\ d &= \frac{\lambda}{4} + 2n\lambda \\ &= (0.103 + 0.822n) \quad (\mu\mathrm{m}), \qquad n = 0,1,2,\ldots\end{aligned}$$

Problem 8.46 A parallel-polarized plane wave is incident from air at an angle $\theta_i = 30^\circ$ onto a pair of dielectric layers as shown in the figure.

(a) Determine the angles of transmission θ_2, θ_3, and θ_4.

(b) Determine the lateral distance *d*.



Solution:

(a) Application of Snell's law of refraction given by (8.31) leads to:

$$\sin\theta_2 = \sin\theta_1\sqrt{\frac{\varepsilon_{r1}}{\varepsilon_{r2}}} = \sin 30^\circ\sqrt{\frac{1}{6.25}} = 0.2$$
$$\theta_2 = 11.54^\circ.$$

Similarly,

$$\sin\theta_3 = \sin\theta_2\sqrt{\frac{\varepsilon_{r2}}{\varepsilon_{r3}}} = \sin 11.54^\circ\sqrt{\frac{6.25}{2.25}} = 0.33$$
$$\theta_3 = 19.48^\circ.$$

And,

$$\sin\theta_4 = \sin\theta_3\sqrt{\frac{\varepsilon_{r3}}{\varepsilon_{r4}}} = \sin 19.48^\circ\sqrt{\frac{2.25}{1}} = 0.5$$
$$\theta_4 = 30^\circ.$$

As expected, the exit ray back into air will be at the same angle as θ_i.

(b)

$$\begin{aligned} d &= (5\text{ cm})\tan\theta_2 + (5\text{ cm})\tan\theta_3 \\ &= 5\tan 11.54^\circ + 5\tan 19.48^\circ = 2.79\text{ cm}. \end{aligned}$$

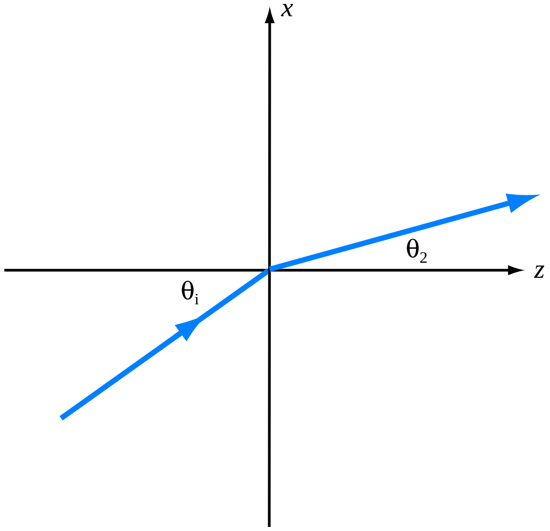
Problem 8.47 A plane wave in air with

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = (\hat{\mathbf{x}}2 - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}6)e^{-j(2x+3z)} \quad \text{(V/m)}$$

is incident upon the planar surface of a dielectric material, with $\varepsilon_{\mathrm{r}} = 2.25$, occupying the half-space $z \geq 0$. Determine

- **(a)** The incidence angle θ_{i}.
- **(b)** The frequency of the wave.
- **(c)** The field $\widetilde{\mathbf{E}}^{\mathrm{r}}$ of the reflected wave.
- **(d)** The field $\widetilde{\mathbf{E}}^{\mathrm{t}}$ of the wave transmitted into the dielectric medium.
- **(e)** The average power density carried by the wave into the dielectric medium.

Solution:



(a) From the exponential of the given expression, it is clear that the wave direction of travel is in the x–z plane. By comparison with the expressions in (8.48a) for perpendicular polarization or (8.65a) for parallel polarization, both of which have the same phase factor, we conclude that:

$$k_1 \sin\theta_\mathrm{i} = 2,$$
$$k_1 \cos\theta_\mathrm{i} = 3.$$

Hence,

$$k_1 = \sqrt{2^2 + 3^2} = 3.6 \quad \text{(rad/m)}$$
$$\theta_\mathrm{i} = \tan^{-1}(2/3) = 33.7^\circ.$$

Also,

$$k_2 = k_1\sqrt{\varepsilon_{\mathrm{r}_2}} = 3.6\sqrt{2.25} = 5.4 \quad \text{(rad/m)}$$
$$\theta_2 = \sin^{-1}\left[\sin\theta_\mathrm{i}\sqrt{\frac{1}{2.25}}\right] = 21.7^\circ.$$

(b)

$$k_1 = \frac{2\pi f}{c}$$
$$f = \frac{k_1 c}{2\pi} = \frac{3.6 \times 3 \times 10^8}{2\pi} = 172 \text{ MHz}.$$

(c) In order to determine the electric field of the reflected wave, we first have to determine the polarization of the wave. The vector argument in the given expression for $\widetilde{\mathbf{E}}^\mathrm{i}$ indicates that the incident wave is a mixture of parallel and perpendicular polarization components. Perpendicular polarization has a $\hat{\mathbf{y}}$-component only (see 8.46a), whereas parallel polarization has only $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components (see 8.65a). Hence, we shall decompose the incident wave accordingly:

$$\widetilde{\mathbf{E}}^\mathrm{i} = \widetilde{\mathbf{E}}^\mathrm{i}_\perp + \widetilde{\mathbf{E}}^\mathrm{i}_\parallel$$

with

$$\widetilde{\mathbf{E}}^\mathrm{i}_\perp = -\hat{\mathbf{y}}\,4e^{-j(2x+3z)} \quad \text{(V/m)}$$
$$\widetilde{\mathbf{E}}^\mathrm{i}_\parallel = (\hat{\mathbf{x}}\,2 - \hat{\mathbf{z}}\,6)e^{-j(2x+3z)} \quad \text{(V/m)}$$

From the above expressions, we deduce:

$$E^\mathrm{i}_{\perp 0} = -4 \text{ V/m}$$
$$E^\mathrm{i}_{\parallel 0} = \sqrt{2^2 + 6^2} = 6.32 \text{ V/m}.$$

Next, we calculate Γ and τ for each of the two polarizations:

$$\Gamma_\perp = \frac{\cos\theta_\mathrm{i} - \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}}{\cos\theta_\mathrm{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}}$$

Using $\theta_\mathrm{i} = 33.7^\circ$ and $\varepsilon_2/\varepsilon_1 = 2.25/1 = 2.25$ leads to:

$$\begin{aligned} \Gamma_\perp &= -0.25 \\ \tau_\perp &= 1 + \Gamma_\perp = 0.75. \end{aligned}$$

Similarly,

$$\begin{aligned} \Gamma_\perp &= \frac{-(\varepsilon_2/\varepsilon_1)\cos\theta_\mathrm{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}}{(\varepsilon_2/\varepsilon_1)\cos\theta_\mathrm{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}} = -0.15, \\ \tau_\parallel &= (1 + \Gamma_\parallel)\frac{\cos\theta_\mathrm{i}}{\cos\theta_\mathrm{t}} = (1 - 0.15)\frac{\cos 33.7^\circ}{\cos 21.7^\circ} = 0.76. \end{aligned}$$

The electric fields of the reflected and transmitted waves for the two polarizations are given by (8.49a), (8.49c), (8.65c), and (8.65e):

$$\begin{aligned} \widetilde{\mathbf{E}}^\mathrm{r}_\perp &= \hat{\mathbf{y}} E^\mathrm{r}_{\perp 0} e^{-jk_1(x\sin\theta_\mathrm{r} - z\cos\theta_\mathrm{r})} \\ \widetilde{\mathbf{E}}^\mathrm{t}_\perp &= \hat{\mathbf{y}} E^\mathrm{t}_{\perp 0} e^{-jk_2(x\sin\theta_\mathrm{t} + z\cos\theta_\mathrm{t})} \\ \widetilde{\mathbf{E}}^\mathrm{r}_\parallel &= (\hat{\mathbf{x}}\cos\theta_\mathrm{r} + \hat{\mathbf{z}}\sin\theta_\mathrm{r}) E^\mathrm{r}_{\parallel 0} e^{-jk_1(x\sin\theta_\mathrm{r} - z\cos\theta_\mathrm{r})} \\ \widetilde{\mathbf{E}}^\mathrm{t}_\parallel &= (\hat{\mathbf{x}}\cos\theta_\mathrm{t} - \hat{\mathbf{z}}\sin\theta_\mathrm{t}) E^\mathrm{t}_{\parallel 0} e^{-jk_2(x\sin\theta_\mathrm{t} + z\cos\theta_\mathrm{t})} \end{aligned}$$

Based on our earlier calculations:

$$\begin{aligned} \theta_\mathrm{r} &= \theta_\mathrm{i} = 33.7^\circ \\ \theta_\mathrm{t} &= 21.7^\circ \\ k_1 &= 3.6 \text{ rad/m}, \qquad k_2 = 5.4 \text{ rad/m}, \\ E^\mathrm{r}_{\perp 0} &= \Gamma_\perp E^\mathrm{i}_{\perp 0} = (-0.25) \times (-4) = 1 \text{ V/m}. \\ E^\mathrm{t}_{\perp 0} &= \tau_\perp E^\mathrm{i}_{\perp 0} = 0.75 \times (-4) = -3 \text{ V/m}. \\ E^\mathrm{r}_{\parallel 0} &= \Gamma_\parallel E^\mathrm{i}_{\parallel 0} = (-0.15) \times 6.32 = -0.95 \text{ V/m}. \\ E^\mathrm{t}_{\parallel 0} &= \tau_\parallel E^\mathrm{i}_{\parallel 0} = 0.76 \times 6.32 = 4.8 \text{ V/m}. \end{aligned}$$

Using the above values, we have:

$$\begin{aligned} \widetilde{\mathbf{E}}^\mathrm{r} &= \widetilde{\mathbf{E}}^\mathrm{r}_\perp + \widetilde{\mathbf{E}}^\mathrm{r}_\parallel \\ &= (-\hat{\mathbf{x}}0.79 + \hat{\mathbf{y}} - \hat{\mathbf{z}}0.53) e^{-j(2x - 3z)} \quad \text{(V/m)}. \end{aligned}$$

(d)

$$\begin{aligned}\widetilde{\mathbf{E}}^{\rm t} &= \widetilde{\mathbf{E}}^{\rm t}_{\perp} + \widetilde{\mathbf{E}}^{\rm t}_{\parallel} \\ &= (\hat{\mathbf{x}}4.46 - \hat{\mathbf{y}}3 - \hat{\mathbf{z}}1.78)e^{-j(2x+5z)} \quad \text{(V/m)}.\end{aligned}$$

(e)

$$\begin{aligned}S^{\rm t} &= \frac{|E_0^{\rm t}|^2}{2\eta_2} \\ |E_0^{\rm t}|^2 &= (4.46)^2 + 3^2 + (1.78)^2 = 32.06 \\ \eta_2 &= \frac{\eta_0}{\sqrt{\varepsilon_{\rm r_2}}} = \frac{377}{1.5} = 251.3\ \Omega \\ S^{\rm t} &= \frac{32.06}{2\times 251.3} = 63.8 \quad \text{(mW/m}^2\text{)}.\end{aligned}$$

Chapter 9: Radiation and Antennas

Lesson #61

Chapter — Section: 9-1

Topics: Retarded potential, short dipole

Highlights:

- Radiation by short dipole
- Far-field distance

Special Illustrations:

- Exercise 9.1

Lesson #62

Chapter — Section: 9-2

Topics: Radiation characteristics

Highlights:

- Antenna pattern
- Antenna directivity
- Antenna gain

Special Illustrations:

- Example 9-2
- Example 9-3

Lesson #63

Chapter — Section: 9-3 and 9-4

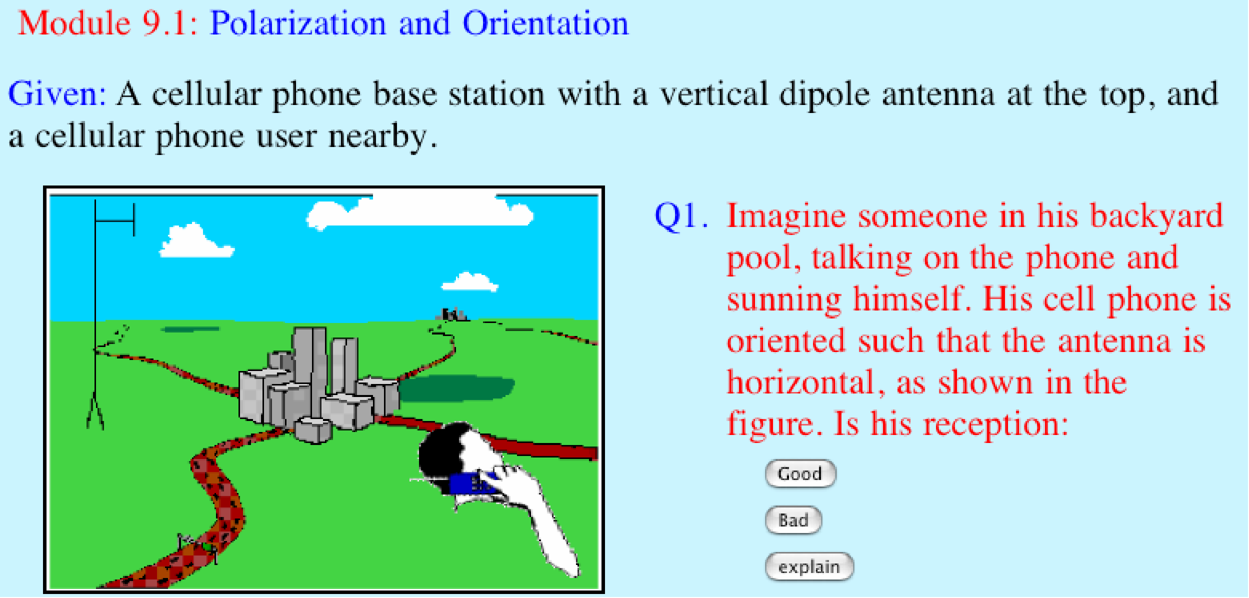
Topics: Half-wave dipole

Highlights:

- Radiation pattern
- Directivity
- Radiation resistance

Special Illustrations:

- CD-ROM Module 9.1
- CD-ROM Demo 9.1



Module 9.1: Polarization and Orientation

Given: A cellular phone base station with a vertical dipole antenna at the top, and a cellular phone user nearby.

Q1. Imagine someone in his backyard pool, talking on the phone and sunning himself. His cell phone is oriented such that the antenna is horizontal, as shown in the figure. Is his reception:

Lesson #64

Chapter — Section: 9-5, 9-6

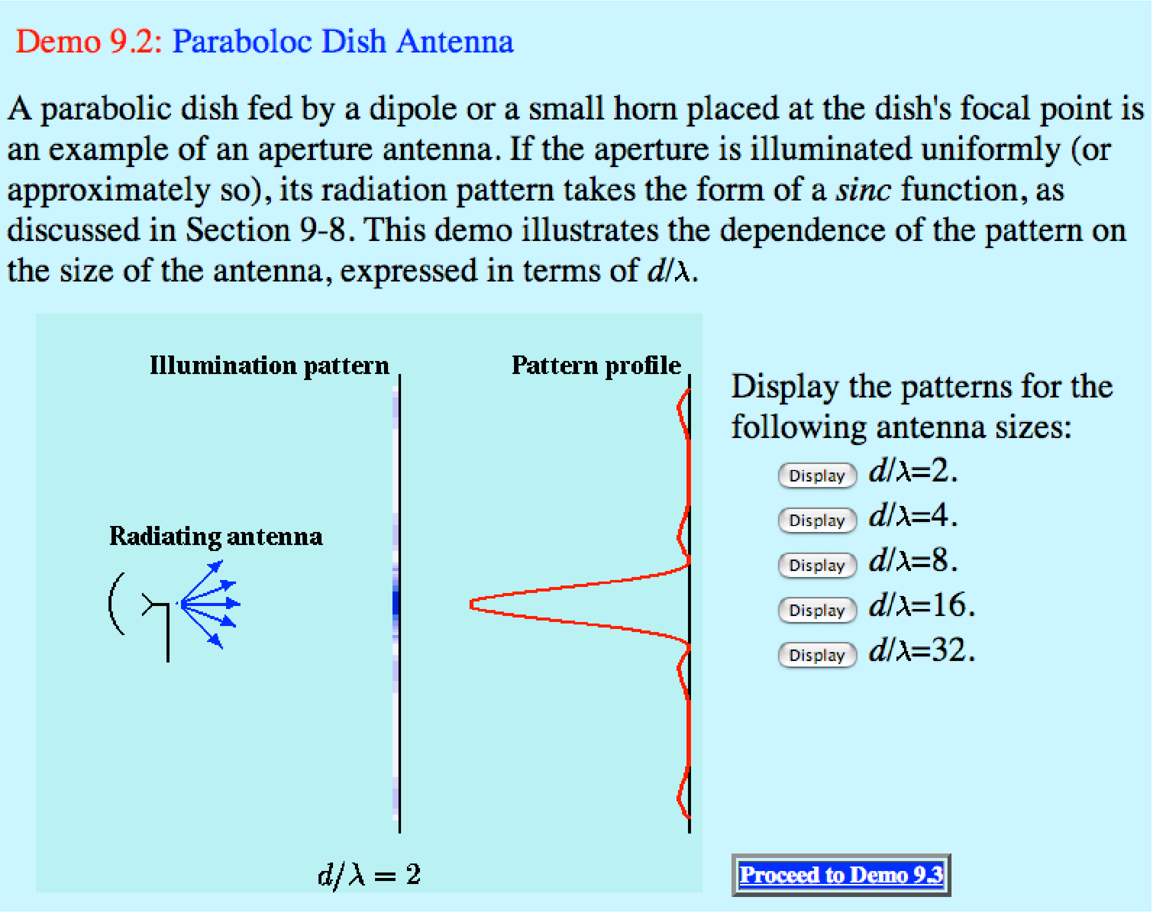
Topics: Effective area, Friis formula

Highlights:

- Receiving aperture of an antenna
- Relation of aperture to directivity
- Friis formula

Special Illustrations:

- Example 9-5
- Demo 9.2



Demo 9.2: Paraboloc Dish Antenna

A parabolic dish fed by a dipole or a small horn placed at the dish's focal point is an example of an aperture antenna. If the aperture is illuminated uniformly (or approximately so), its radiation pattern takes the form of a *sinc* function, as discussed in Section 9-8. This demo illustrates the dependence of the pattern on the size of the antenna, expressed in terms of d/λ.

Display the patterns for the following antenna sizes:

Display $d/\lambda=2$.
Display $d/\lambda=4$.
Display $d/\lambda=8$.
Display $d/\lambda=16$.
Display $d/\lambda=32$.

Lessons #65 and 66

Chapter — Sections: 9-7 and 9-8

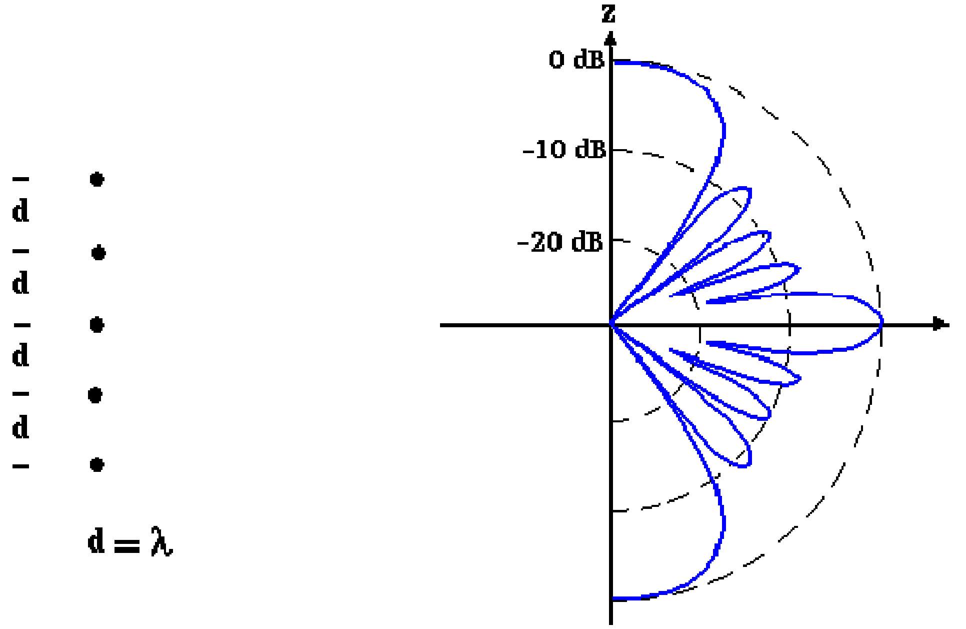
Topics: Aperture antennas

Highlights:

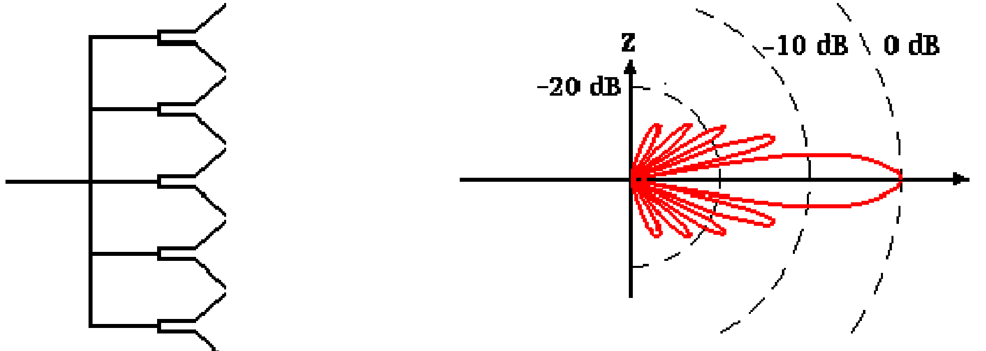
- Aperture illumination
- Rectangular aperture
- Beamwidth and directivity

Special Illustrations:

- CD-ROM Demo 9.3



(b) Array Pattern $F_a(\theta)$



(c) Antenna Array Pattern $F(\theta) = F_e(\theta) F_a(\theta)$

Lessons #67–69

Chapter — Sections: 9-9 to 9-11

Topics: Antenna arrays

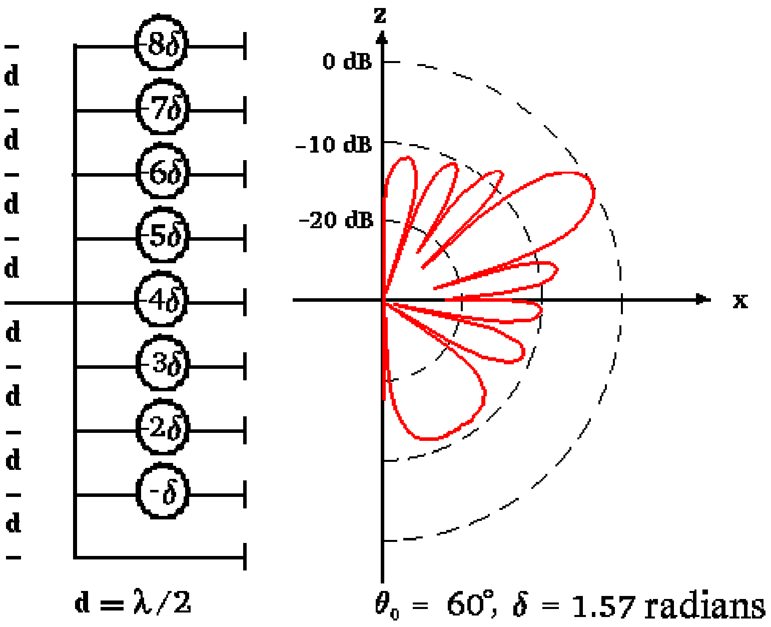
Highlights:

- Array factor
- Multiplication principle
- Electronic scanning

Special Illustrations:

- CD-ROM Demo 9.4

The array pattern of an equally-spaced linear array can be steered in direction by applying linear phase across the array as shown. Note that $\delta = kd \cos \theta_0$, with θ_0 measured from the $+z$-axis.



Display the array pattern for the following values of the beam center angle:

$\theta_0 = 90^\circ$ (broadside)

$\theta_0 = 60^\circ$
(30° above x-axis)

$\theta_0 = 30^\circ$
(60° above x-axis)

$\theta_0 = 120^\circ$
(30° below x-axis)

$\theta_0 = 150^\circ$
(60° below x-axis)

Chapter 9

Sections 9-1 and 9-2: Short Dipole and Antenna Radiation Characteristics

Problem 9.1 A center-fed Hertzian dipole is excited by a current $I_0 = 20$ A. If the dipole is $\lambda/50$ in length, determine the maximum radiated power density at a distance of 1 km.

Solution: From Eq. (9.14), the maximum power density radiated by a Hertzian dipole is given by

$$S_0 = \frac{\eta_0 k^2 I_0^2 l^2}{32\pi^2 R^2} = \frac{377 \times (2\pi/\lambda)^2 \times 20^2 \times (\lambda/50)^2}{32\pi^2 (10^3)^2}$$
$$= 7.6 \times 10^{-6} \text{ W/m}^2 = 7.6 \quad (\mu\text{W/m}^2).$$

Problem 9.2 A 1-m-long dipole is excited by a 1-MHz current with an amplitude of 12 A. What is the average power density radiated by the dipole at a distance of 5 km in a direction that is 45° from the dipole axis?

Solution: At 1 MHz, $\lambda = c/f = 3 \times 10^8/10^6 = 300$ m. Hence $l/\lambda = 1/300$, and therefore the antenna is a Hertzian dipole. From Eq. (9.12),

$$S(R,\theta) = \left(\frac{\eta_0 k^2 I_0^2 l^2}{32\pi^2 R^2}\right) \sin^2\theta$$
$$= \frac{120\pi \times (2\pi/300)^2 \times 12^2 \times 1^2}{32\pi^2 \times (5 \times 10^3)^2} \sin^2 45^\circ = 1.51 \times 10^{-9} \quad (\text{W/m}^2).$$

Problem 9.3 Determine the (a) direction of maximum radiation, (b) directivity, (c) beam solid angle, and (d) half-power beamwidth in the x–z plane for an antenna whose normalized radiation intensity is given by

$$F(\theta,\phi) = \begin{cases} 1, & \text{for } 0 \le \theta \le 60^\circ \text{and } 0 \le \phi \le 2\pi, \\ 0, & \text{elsewhere.} \end{cases}$$

Suggestion: Sketch the pattern prior to calculating the desired quantities.

Solution: The direction of maximum radiation is a circular cone 120° wide centered around the $+\hat{\mathbf{z}}$-axis. From Eq. (9.23),

$$D = \frac{4\pi}{\iint_{4\pi} F \, d\Omega} = \frac{4\pi}{\int_0^{2\pi} \int_0^{60^\circ} \sin\theta \, d\theta \, d\phi} = \frac{4\pi}{2\pi(-\cos\theta)\big|_{0^\circ}^{60^\circ}} = \frac{2}{-\frac{1}{2}+1} = 4 = 6 \text{ dB},$$

$$\Omega_\text{p} = \frac{4\pi \text{ sr}}{D} = \frac{4\pi \text{ sr}}{4} = \pi \quad \text{(sr)}.$$

The half power beamwidth is $\beta = 120°$.

Problem 9.4 Repeat Problem 9.3 for an antenna with

$$F(\theta,\phi) = \begin{cases} \sin^2\theta\cos^2\phi, & \text{for } 0 \le \theta \le \pi \text{ and } -\pi/2 \le \phi \le \pi/2, \\ 0, & \text{elsewhere.} \end{cases}$$

Solution: The direction of maximum radiation is the $+\hat{\mathbf{x}}$-axis (where $\theta = \pi/2$ and $\phi = 0$). From Eq. (9.23),

$$\begin{aligned} D &= \frac{4\pi}{\iint_{4\pi} F \, d\Omega} \\ &= \frac{4\pi}{\int_{-\pi/2}^{\pi/2} \int_0^\pi \sin^2\theta \cos^2\phi \sin\theta \, d\theta \, d\phi} \\ &= \frac{4\pi}{\int_{-\pi/2}^{\pi/2} \cos^2\phi d\phi \int_0^\pi \sin^3\theta \, d\theta} \\ &= \frac{4\pi}{\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos 2\phi)\, d\phi \int_{-1}^{1} (1-x^2)\, dx} \\ &= \frac{4\pi}{\frac{1}{2}\left(\phi + \frac{1}{2}\sin 2\phi\right)\Big|_{-\pi/2}^{\pi/2} (x - x^3/3)\Big|_{-1}^{1}} = \frac{4\pi}{\frac{1}{2}\pi(4/3)} = 6 = 7.8 \text{ dB}, \\ \Omega_\text{p} &= \frac{4\pi \text{ sr}}{D} = \frac{4\pi \text{ sr}}{6} = \frac{2}{3}\pi \quad \text{(sr)}. \end{aligned}$$

In the x-z plane, $\phi = 0$ and the half power beamwidth is 90°, since $\sin^2(45°) = \sin^2(135°) = \frac{1}{2}$.

Problem 9.5 A 2-m-long center-fed dipole antenna operates in the AM broadcast band at 1 MHz. The dipole is made of copper wire with a radius of 1 mm.

(a) Determine the radiation efficiency of the antenna.

(b) What is the antenna gain in dB?

(c) What antenna current is required so that the antenna would radiate 80 W, and how much power will the generator have to supply to the antenna?

Solution:

(a) Following Example 9-3, $\lambda = c/f = (3 \times 10^8 \text{ m/s})/(10^6 \text{ Hz}) = 300$ m. As $l/\lambda = (2 \text{ m})/(300 \text{ m}) = 6.7 \times 10^{-3}$, this antenna is a short (Hertzian) dipole. Thus, from respectively Eqs. (9.35), (9.32), and (9.31),

$$R_{\text{rad}} = 80\pi^2 \left(\frac{l}{\lambda}\right)^2 = 80\pi^2 (6.7 \times 10^{-3})^2 = 35 \quad (\text{m}\Omega),$$

$$R_{\text{loss}} = \frac{l}{2\pi a}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \frac{2 \text{ m}}{2\pi(10^{-3} \text{ m})}\sqrt{\frac{\pi(10^6 \text{ Hz})(4\pi \times 10^{-7} \text{ H/m})}{5.8 \times 10^7 \text{ S/m}}} = 83 \quad (\text{m}\Omega),$$

$$\xi = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_{\text{loss}}} = \frac{35 \text{ m}\Omega}{35 \text{ m}\Omega + 83 \text{ m}\Omega} = 29.7\%.$$

(b) From Example 9-2, a Hertzian dipole has a directivity of 1.5. The gain, from Eq. (9.29), is $G = \xi D = 0.297 \times 1.5 = 0.44 = -3.5$ dB.

(c) From Eq. (9.30a),

$$I_0 = \sqrt{\frac{2P_{\text{rad}}}{R_{\text{rad}}}} = \sqrt{\frac{2(80 \text{ W})}{35 \text{ m}\Omega}} = 67.6 \text{ A}$$

and from Eq. (9.31),

$$P_t = \frac{P_{\text{rad}}}{\xi} = \frac{80 \text{ W}}{0.297} = 269 \text{ W}.$$

Problem 9.6 Repeat Problem 9.5 for a 20-cm-long antenna operating at 5 MHz.

Solution:

(a) At 5 MHz, $\lambda = c/f = 3 \times 10^8/(5 \times 10^6) = 60$ m. As $l/\lambda = 0.2/60 = 3.33 \times 10^{-3}$, the antenna length satisfies the condition of a short dipole. From Eqs. (9.35), (9.32), and (9.31),

$$R_{\text{rad}} = 80\pi^2 \left(\frac{l}{\lambda}\right)^2 = 80\pi^2 \times (3.33 \times 10^{-3})^2 = 8.76 \quad (\text{m}\Omega),$$

$$R_{\text{loss}} = \frac{l}{2\pi a}\sqrt{\frac{\pi f u_c}{\sigma_c}} = \frac{0.2}{2\pi \times 10^{-3}}\sqrt{\frac{\pi \times 5 \times 10^6 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} = 18.57 \quad (\text{m}\Omega),$$

$$\xi = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_{\text{loss}}} = \frac{8.76}{8.76 + 18.57} = 0.32, \quad \text{or } 32\%.$$

(b) For Hertzian dipole, $D = 1.5$, and $G = \xi D = 0.32 \times 1.5 = 0.48 = -3.2$dB.

(c) From Eq. (9.30a),

$$I_0 = \sqrt{\frac{2P_{\text{rad}}}{R_{\text{rad}}}} = \sqrt{\frac{2 \times 80}{8.76 \times 10^{-3}}} = 135.2 \text{ A}.$$

Problem 9.7 An antenna with a pattern solid angle of 1.5 (sr) radiates 60 W of power. At a range of 1 km, what is the maximum power density radiated by the antenna?

Solution: From Eq. (9.23), $D = 4\pi/\Omega_{\rm p}$, and from Eq. (9.24), $D = 4\pi R^2 S_{\rm max}/P_{\rm rad}$. Combining these two equations gives

$$S_{\rm max} = \frac{P_{\rm rad}}{\Omega_{\rm p} R^2} = \frac{60}{1.5 \times (10^3)^2} = 4 \times 10^{-5} \quad ({\rm W/m^2}).$$

Problem 9.8 An antenna with a radiation efficiency of 90% has a directivity of 7.0 dB. What is its gain in dB?

Solution: $D = 7.0$ dB corresponds to $D = 5.0$.

$$G = \xi D = 0.9 \times 5.0 = 4.5 = 6.54 \text{ dB}.$$

Alternatively,

$$G\ ({\rm dB}) = \xi\ ({\rm dB}) + D\ ({\rm dB}) = 10 \log 0.9 + 7.0 = -0.46 + 7.0 = 6.54 \text{ dB}.$$

Problem 9.9 The radiation pattern of a circular parabolic-reflector antenna consists of a circular major lobe with a half-power beamwidth of 3° and a few minor lobes. Ignoring the minor lobes, obtain an estimate for the antenna directivity in dB.

Solution: A circular lobe means that $\beta_{xz} = \beta_{yz} = 3^\circ = 0.052$ rad. Using Eq. (9.26), we have

$$D = \frac{4\pi}{\beta_{xz}\beta_{yz}} = \frac{4\pi}{(0.052)^2} = 4.58 \times 10^3.$$

In dB,

$$D({\rm dB}) = 10 \log D = 10 \log(4.58 \times 10^3) = 36.61 \text{ dB}.$$

Problem 9.10 The normalized radiation intensity of a certain antenna is given by

$$F(\theta) = \exp(-20\theta^2) \qquad \text{for } 0 \le \theta \le \pi,$$

where θ is in radians. Determine:

(a) the half-power beamwidth,

(b) the pattern solid angle,

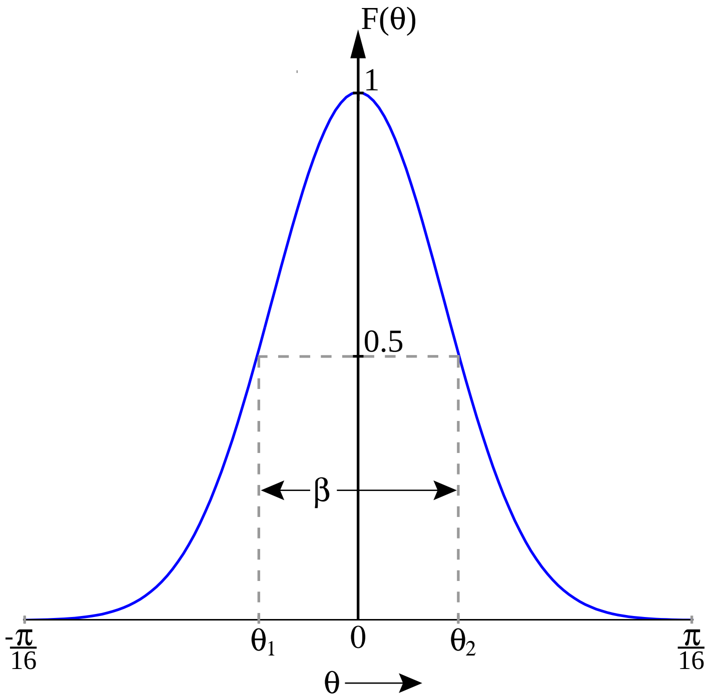


Figure P9.10: $F(\theta)$ versus θ.

(c) the antenna directivity.

Solution:

(a) Since $F(\theta)$ is independent of ϕ, the beam is symmetrical about $z = 0$. Upon setting $F(\theta) = 0.5$, we have

$$\begin{aligned} F(\theta) &= \exp(-20\theta^2) = 0.5, \\ \ln[\exp(-20\theta^2)] &= \ln(0.5), \\ 20\theta^2 &= -0.693, \\ \theta &= \pm\left(\frac{0.693}{20}\right)^{1/2} = \pm 0.186 \text{ radians}. \end{aligned}$$

Hence, $\beta = 2 \times 0.186 = 0.372$ radians $= 21.31°$.

(b) By Eq. (9.21),

$$\begin{aligned}\Omega_p &= \iint_{4\pi} F(\theta)\sin\theta\,d\theta\,d\phi \\ &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \exp(-20\theta^2)\sin\theta\,d\theta\,d\phi \\ &= 2\pi\int_0^{\pi} \exp(-20\theta^2)\sin\theta\,d\theta.\end{aligned}$$

Numerical evaluation yields

$$\Omega_p = 0.156 \text{ sr}.$$

(c)

$$D = \frac{4\pi}{\Omega_p} = \frac{4\pi}{0.156} = 80.55.$$

Sections 9-3 and 9-4: Dipole Antennas

Problem 9.11 Repeat Problem 9.5 for a 1-m-long half-wave dipole that operates in the FM/TV broadcast band at 150 MHz.

Solution:

(a) Following Example 9-3,

$$\lambda = c/f = (3\times10^8 \text{ m/s})/(150\times10^6 \text{ Hz}) = 2 \text{ m}.$$

As $l/\lambda = (1\text{ m})/(2\text{ m}) = \frac{1}{2}$, this antenna is a half-wave dipole. Thus, from Eq. (9.48), (9.32), and (9.31),

$$\begin{aligned}R_{\text{rad}} &= 73\ \Omega, \\ R_{\text{loss}} &= \frac{l}{2\pi a}\sqrt{\frac{\pi f\mu_c}{\sigma_c}} = \frac{1\text{ m}}{2\pi(10^{-3}\text{ m})}\sqrt{\frac{\pi(150\times10^6\text{ Hz})(4\pi\times10^{-7}\text{ H/m})}{5.8\times10^7\text{ S/m}}} = 0.5\ \Omega, \\ \xi &= \frac{R_{\text{rad}}}{R_{\text{rad}}+R_{\text{loss}}} = \frac{73\ \Omega}{73\ \Omega+0.5\ \Omega} = 99.3\%.\end{aligned}$$

(b) From Eq. (9.47), a half-wave dipole has a directivity of 1.64. The gain, from Eq. (9.29), is $G = \xi D = 0.993\times1.64 = 1.63 = 2.1$ dB.

(c) From Eq. (9.30a),

$$I_0 = \sqrt{\frac{2P_{\text{rad}}}{R_{\text{rad}}}} = \sqrt{\frac{2(80\text{ W})}{73\ \Omega}} = 1.48 \text{ A},$$

and from Eq. (9.31),

$$P_{\mathrm{t}} = \frac{P_{\mathrm{rad}}}{\xi} = \frac{80\ \mathrm{W}}{0.993} = 80.4\ \mathrm{W}.$$

Problem 9.12 Assuming the loss resistance of a half-wave dipole antenna to be negligibly small and ignoring the reactance component of its antenna impedance, calculate the standing wave ratio on a 50-Ω transmission line connected to the dipole antenna.

Solution: According to Eq. (9.48), a half wave dipole has a radiation resistance of 73 Ω. To the transmission line, this behaves as a load, so the reflection coefficient is

$$\Gamma = \frac{R_{\mathrm{rad}} - Z_0}{R_{\mathrm{rad}} + Z_0} = \frac{73\ \Omega - 50\ \Omega}{73\ \Omega + 50\ \Omega} = 0.187,$$

and the standing wave ratio is

$$S = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.187}{1-0.187} = 1.46.$$

Problem 9.13 For the short dipole with length l such that $l \ll \lambda$, instead of treating the current $\widetilde{I}(z)$ as constant along the dipole, as was done in Section 9-1, a more realistic approximation that insures that the current goes to zero at the ends is to describe $\widetilde{I}(z)$ by the triangular function

$$\widetilde{I}(z) = \begin{cases} I_0(1-2z/l), & \text{for } 0 \le z \le l/2, \\ I_0(1+2z/l), & \text{for } -l/2 \le z \le 0, \end{cases}$$

as shown in Fig. 9-36 (P9.13). Use this current distribution to determine (a) the far-field $\widetilde{E}(R,\theta,\phi)$, (b) the power density $S(R,\theta,\phi)$, (c) the directivity D, and (d) the radiation resistance R_{rad}.

Solution:

(a) When the current along the dipole was assumed to be constant and equal to I_0, the vector potential was given by Eq. (9.3) as:

$$\widetilde{\mathbf{A}}(R) = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\left(\frac{e^{-jkR}}{R}\right)\int_{-l/2}^{l/2} I_0\, dz.$$

If the triangular current function is assumed instead, then I_0 in the above expression should be replaced with the given expression. Hence,

$$\widetilde{\mathbf{A}} = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\left(\frac{e^{-jkR}}{R}\right) I_0 \left[\int_0^{l/2}\left(1-\frac{2z}{l}\right)dz + \int_{-l/2}^{0}\left(1+\frac{2z}{l}\right)dz\right] = \hat{\mathbf{z}}\frac{\mu_0 I_0 l}{8\pi}\left(\frac{e^{-jkR}}{R}\right),$$

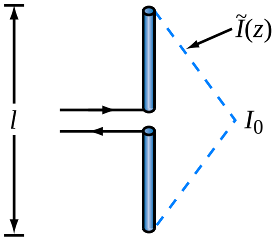


Figure P9.13: Triangular current distribution on a short dipole (Problem 9.13).

which is half that obtained for the constant-current case given by Eq. (9.3). Hence, the expression given by (9.9a) need only be modified by the factor of $1/2$:

$$\widetilde{\mathbf{E}} = \hat{\boldsymbol{\theta}}\widetilde{E}_\theta = \hat{\boldsymbol{\theta}}\,\frac{jI_0 lk\eta_0}{8\pi}\left(\frac{e^{-jkR}}{R}\right)\sin\theta.$$

(b) The corresponding power density is

$$S(R,\theta) = \frac{|\widetilde{E}_\theta|^2}{2\eta_0} = \left(\frac{\eta_0 k^2 I_0^2 l^2}{128\pi^2 R^2}\right)\sin^2\theta.$$

(c) The power density is 4 times smaller than that for the constant current case, but the reduction is true for all directions. Hence, D remains unchanged at 1.5.

(d) Since $S(R,\theta)$ is 4 times smaller, the total radiated power P_{rad} is 4-times smaller. Consequently, $R_{\text{rad}} = 2P_{\text{rad}}/I_0^2$ is 4 times smaller than the expression given by Eq. (9.35); that is,

$$R_{\text{rad}} = 20\pi^2(l/\lambda)^2 \quad (\Omega).$$

Problem 9.14 For a dipole antenna of length $l = 3\lambda/2$, (a) determine the directions of maximum radiation, (b) obtain an expression for $S_{\max}$, and (c) generate a plot of the normalized radiation pattern $F(\theta)$. Compare your pattern with that shown in Fig. 9.17(c).

Solution:

(a) From Eq. (9.56), $S(\theta)$ for an arbitrary length dipole is given by

$$S(\theta) = \frac{15I_0^2}{\pi R^2}\left[\frac{\cos\left(\frac{\pi l}{\lambda}\cos\theta\right) - \cos\left(\frac{\pi l}{\lambda}\right)}{\sin\theta}\right]^2.$$

For $l = 3\lambda/2$, $S(\theta)$ becomes

$$S(\theta) = \frac{15I_0^2}{\pi R^2}\left[\frac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2.$$

Solving for the directions of maximum radiation numerically yields two maximum directions of radiation given by

$$\theta_{\text{max}_1} = 42.6^\circ, \qquad \theta_{\text{max}_2} = 137.4^\circ.$$

(b) From the numerical results, it was found that $S(\theta) = 15I_0^2/(\pi R^2)(1.96)$ at θ_{max}. Thus,

$$S_{\text{max}} = \frac{15I_0^2}{\pi R^2}(1.96).$$

(c) The normalized radiation pattern is given by Eq. (9.13) as

$$F(\theta) = \frac{S(\theta)}{S_{\text{max}}}.$$

Using the expression for $S(\theta)$ from part (a) with the value of S_{max} found in part (b),

$$F(\theta) = \frac{1}{1.96}\left[\frac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2.$$

The normalized radiation pattern is shown in Fig. P9.14, which is identical to that shown in Fig. 9.17(c).

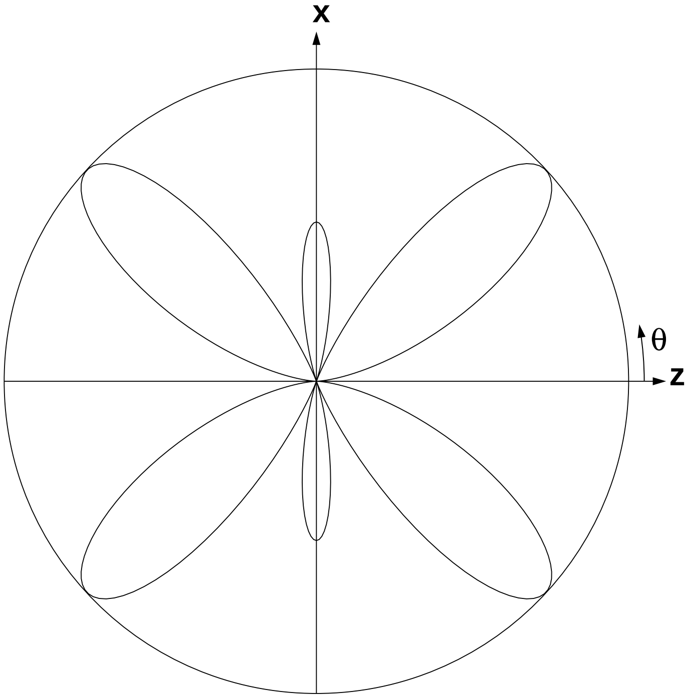


Figure P9.14: Radiation pattern of dipole of length $3\lambda/2$.

Problem 9.15 Repeat parts (a)–(c) of Problem 9.14 for a dipole of length $l = 3\lambda/4$.

Solution:

(a) For $l = 3\lambda/4$, Eq. (9.56) becomes

$$\begin{aligned} S(\theta) &= \frac{15I_0^2}{\pi R^2}\left[\frac{\cos\left(\frac{3\pi}{4}\cos\theta\right) - \cos\left(\frac{3\pi}{4}\right)}{\sin\theta}\right]^2 \\ &= \frac{15I_0^2}{\pi R^2}\left[\frac{\cos\left(\frac{3\pi}{4}\cos\theta\right) + \frac{1}{\sqrt{2}}}{\sin\theta}\right]^2 . \end{aligned}$$

Solving for the directions of maximum radiation numerically yields

$$\theta_{\max_1} = 90^\circ, \qquad \theta_{\max_2} = 270^\circ.$$

(b) From the numerical results, it was found that $S(\theta) = 15I_0^2/(\pi R^2)(2.91)$ at $\theta_{\max}$.

Thus,

$$S_{\max} = \frac{15 I_0^2}{\pi R^2}\,(2.91).$$

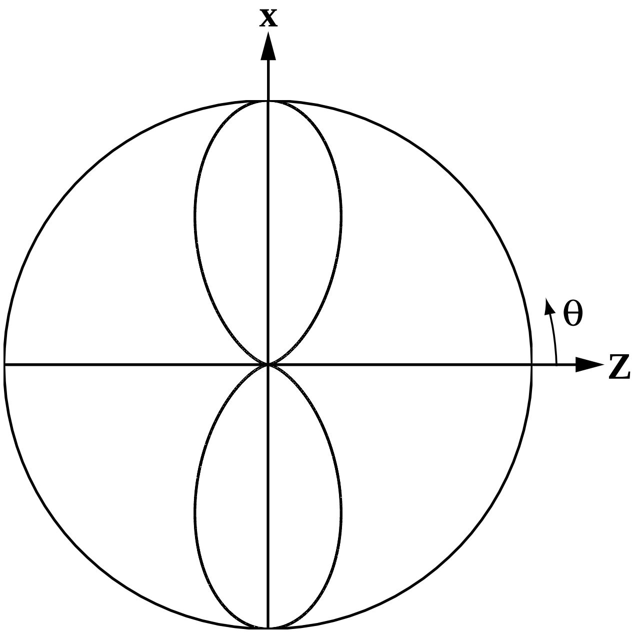


Figure P9.15: Radiation pattern of dipole of length $l = 3\lambda/4$.

(c) The normalized radiation pattern is given by Eq. (9.13) as

$$F(\theta) = \frac{S(\theta)}{S_{\max}}.$$

Using the expression for $S(\theta)$ from part (a) with the value of $S_{\max}$ found in part (b),

$$F(\theta) = \frac{1}{2.91}\left[\frac{\cos\left(\frac{3\pi}{4}\cos\theta\right) + \frac{1}{\sqrt{2}}}{\sin\theta}\right]^2.$$

The normalized radiation pattern is shown in Fig. P9.15.

Problem 9.16 Repeat parts (a)–(c) of Problem 9.14 for a dipole of length $l = \lambda$.

Solution: For $l = \lambda$, Eq. (9.56) becomes

$$S(\theta) = \frac{15I_0^2}{\pi R^2}\left[\frac{\cos(\pi\cos\theta) - \cos(\pi)}{\sin\theta}\right]^2 = \frac{15I_0^2}{\pi R^2}\left[\frac{\cos(\pi\cos\theta) + 1}{\sin\theta}\right]^2.$$

Solving for the directions of maximum radiation numerically yields

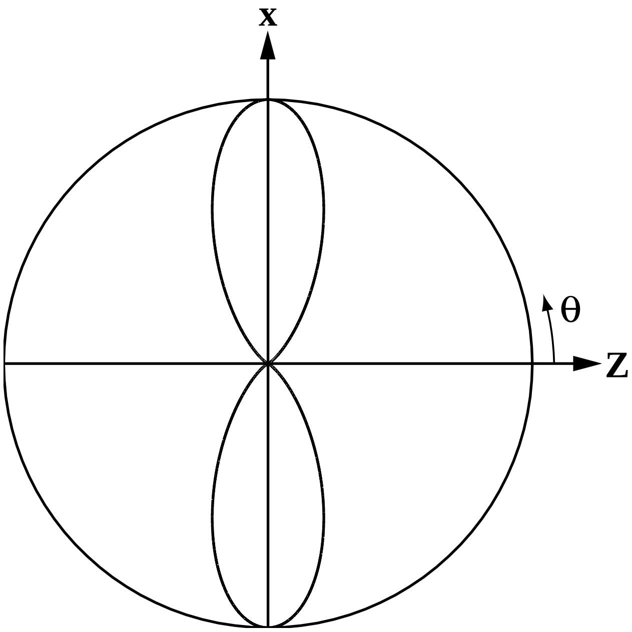


Figure P9.16: Radiation pattern of dipole of length $l = \lambda$.

$$\theta_{\max_1} = 90^\circ, \qquad \theta_{\max_2} = 270^\circ.$$

(b) From the numerical results, it was found that $S(\theta) = 15I_0^2/(\pi R^2)(4)$ at $\theta_{\max}$. Thus,

$$S_{\max} = \frac{60I_0^2}{\pi R^2}.$$

(c) The normalized radiation pattern is given by Eq. (9.13), as

$$F(\theta) = \frac{S(\theta)}{S_{\max}}.$$

Using the expression for $S(\theta)$ from part (a) with the value of $S_{\max}$ found in part (b),

$$F(\theta) = \frac{1}{4}\left[\frac{\cos(\pi\cos\theta) + 1}{\sin\theta}\right]^2.$$

The normalized radiation pattern is shown in Fig. P9.16.

Problem 9.17 A car antenna is a vertical monopole over a conducting surface. Repeat Problem 9.5 for a 1-m-long car antenna operating at 1 MHz. The antenna wire is made of aluminum with $\mu_c = \mu_0$ and $\sigma_c = 3.5 \times 10^7$ S/m, and its diameter is 1 cm.

Solution:

(a) Following Example 9-3, $\lambda = c/f = (3 \times 10^8 \text{ m/s})/(10^6 \text{ Hz}) = 300$ m. As $l/\lambda = 2 \times (1 \text{ m})/(300 \text{ m}) = 0.0067$, this antenna is a short (Hertzian) monopole. From Section 9-3.3, the radiation resistance of a monopole is half that for a corresponding dipole. Thus,

$$R_{\text{rad}} = \tfrac{1}{2} 80\pi^2 \left(\frac{l}{\lambda}\right)^2 = 40\pi^2 (0.0067)^2 = 17.7 \quad (\text{m}\Omega),$$

$$R_{\text{loss}} = \frac{l}{2\pi a}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \frac{1 \text{ m}}{\pi(10^{-2} \text{ m})}\sqrt{\frac{\pi(10^6 \text{ Hz})(4\pi \times 10^{-7} \text{ H/m})}{3.5 \times 10^7 \text{ S/m}}} = 10.7 \text{ m}\Omega,$$

$$\xi = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_{\text{loss}}} = \frac{17.7 \text{ m}\Omega}{17.7 \text{ m}\Omega + 10.7 \text{ m}\Omega} = 62\%.$$

(b) From Example 9-2, a Hertzian dipole has a directivity of 1.5. The gain, from Eq. (9.29), is $G = \xi D = 0.62 \times 1.5 = 0.93 = -0.3$ dB.

(c) From Eq. (9.30a),

$$I_0 = \sqrt{\frac{2P_{\text{rad}}}{R_{\text{rad}}}} = \sqrt{\frac{2(80 \text{ W})}{17.7 \text{ m}\Omega}} = 95 \text{ A},$$

and from Eq. (9.31),

$$P_t = \frac{P_{\text{rad}}}{\xi} = \frac{80 \text{ W}}{0.62} = 129.2 \text{ W}.$$

Sections 9-5 and 9-6: Effective Area and Friis Formula

Problem 9.18 Determine the effective area of a half-wave dipole antenna at 100 MHz, and compare it to its physical cross section if the wire diameter is 2 cm.

Solution: At $f = 100$ MHz, $\lambda = c/f = (3 \times 10^8 \text{ m/s})/(100 \times 10^6 \text{ Hz}) = 3$ m. From Eq. (9.47), a half wave dipole has a directivity of $D = 1.64$. From Eq. (9.64), $A_e = \lambda^2 D/4\pi = (3 \text{ m})^2 \times 1.64/4\pi = 1.17 \text{ m}^2$.

The physical cross section is: $A_\mathrm{p} = ld = \frac{1}{2}\lambda d = \frac{1}{2}(3\text{ m})(2\times 10^{-2}\text{ m}) = 0.03\text{ m}^2$. Hence, $A_\mathrm{e}/A_\mathrm{p} = 39$.

Problem 9.19 A 3-GHz line-of-sight microwave communication link consists of two lossless parabolic dish antennas, each 1 m in diameter. If the receive antenna requires 10 nW of receive power for good reception and the distance between the antennas is 40 km, how much power should be transmitted?

Solution: At $f = 3$ GHz, $\lambda = c/f = (3\times 10^8\text{ m/s})/(3\times 10^9\text{ Hz}) = 0.10$ m. Solving the Friis transmission formula (Eq. (9.75)) for the transmitted power:

$$\begin{aligned}P_\mathrm{t} &= P_\mathrm{rec}\frac{\lambda^2 R^2}{\xi_\mathrm{t}\xi_\mathrm{r}A_\mathrm{t}A_\mathrm{r}}\\ &= 10^{-8}\frac{(0.100\text{ m})^2(40\times 10^3\text{ m})^2}{1\times 1\times(\frac{\pi}{4}(1\text{ m})^2)(\frac{\pi}{4}(1\text{ m})^2)} = 25.9\times 10^{-2}\text{ W} = 259\text{ mW}.\end{aligned}$$

Problem 9.20 A half-wave dipole TV broadcast antenna transmits 1 kW at 50 MHz. What is the power received by a home television antenna with 3-dB gain if located at a distance of 30 km?

Solution: At $f = 50$ MHz, $\lambda = c/f = (3\times 10^8\text{ m/s})/(50\times 10^6\text{ Hz}) = 6$ m, for which a half wave dipole, or larger antenna, is very reasonable to construct. Assuming the TV transmitter to have a vertical half wave dipole, its gain in the direction of the home would be $G_\mathrm{t} = 1.64$. The home antenna has a gain of $G_\mathrm{r} = 3\text{ dB} = 2$. From the Friis transmission formula (Eq. (9.75)):

$$P_\mathrm{rec} = P_\mathrm{t}\frac{\lambda^2 G_\mathrm{r}G_\mathrm{t}}{(4\pi)^2R^2} = 10^3\frac{(6\text{ m})^2\times 1.64\times 2}{(4\pi)^2(30\times 10^3\text{ m})^2} = 8.3\times 10^{-7}\text{ W}.$$

Problem 9.21 A 150-MHz communication link consists of two vertical half-wave dipole antennas separated by 2 km. The antennas are lossless, the signal occupies a bandwidth of 3 MHz, the system noise temperature of the receiver is 600 K, and the desired signal-to-noise ratio is 17 dB. What transmitter power is required?

Solution: From Eq. (9.77), the receiver noise power is

$$P_\mathrm{n} = KT_\mathrm{sys}B = 1.38\times 10^{-23}\times 600\times 3\times 10^6 = 2.48\times 10^{-14}\text{ W}.$$

For a signal to noise ratio $S_\mathrm{n} = 17\text{ dB} = 50$, the received power must be at least

$$P_\mathrm{rec} = S_\mathrm{n}P_\mathrm{n} = 50(2.48\times 10^{-14}\text{ W}) = 1.24\times 10^{-12}\text{ W}.$$

Since the two antennas are half-wave dipoles, Eq. (9.47) states $D_{\rm t} = D_{\rm r} = 1.64$, and since the antennas are both lossless, $G_{\rm t} = D_{\rm t}$ and $G_{\rm r} = D_{\rm r}$. Since the operating frequency is $f = 150$ MHz, $\lambda = c/f = (3\times10^8 \text{ m/s})/(150\times10^6 \text{ Hz}) = 2$ m. Solving the Friis transmission formula (Eq. (9.75)) for the transmitted power:

$$P_{\rm t} = P_{\rm rec}\frac{(4\pi)^2R^2}{\lambda^2 G_{\rm r}G_{\rm t}} = 1.24\times10^{-12}\frac{(4\pi)^2(2\times10^3 \text{ m})^2}{(2 \text{ m})^2(1.64)(1.64)} = 75 \quad (\mu\text{W}).$$

Problem 9.22 Consider the communication system shown in Fig. 9-37 (P9.22), with all components properly matched. If $P_{\rm t} = 10$ W and $f = 6$ GHz:

(a) what is the power density at the receiving antenna (assuming proper alignment of antennas)?

(b) What is the received power?

(c) If $T_{\rm sys} = 1{,}000$ K and the receiver bandwidth is 20 MHz, what is the signal to noise ratio in dB?

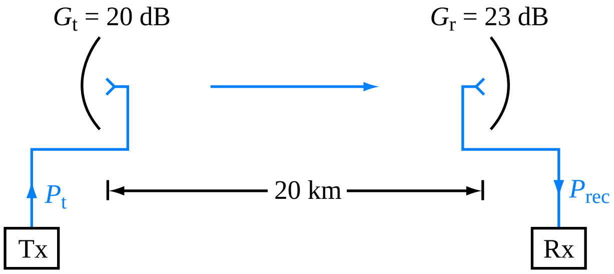


Figure P9.22: Communication system of Problem 9.22.

Solution:

(a) $G_{\rm t} = 20 \text{ dB} = 100$, $G_{\rm r} = 23 \text{ dB} = 200$, and $\lambda = c/f = 5$ cm. From Eq. (9.72),

$$S_{\rm r} = G_{\rm t}\frac{P_{\rm t}}{4\pi R^2} = \frac{10^2\times10}{4\pi\times(2\times10^4)^2} = 2\times10^{-7} \quad (\text{W/m}^2).$$

(b)

$$P_{\rm rec} = P_{\rm t}G_{\rm t}G_{\rm r}\left(\frac{\lambda}{4\pi R}\right)^2 = 10\times100\times200\times\left(\frac{5\times10^{-2}}{4\pi\times2\times10^4}\right)^2 = 7.92\times10^{-9} \text{ W}.$$

(c)

$$P_{\rm n} = KT_{\rm sys}B = 1.38\times10^{-23}\times10^3\times2\times10^7 = 2.76\times10^{-13} \text{ W},$$

$$S_{\text{n}} = \frac{P_{\text{rec}}}{P_{\text{n}}} = \frac{7.92 \times 10^{-9}}{2.76 \times 10^{-13}} = 2.87 \times 10^4 = 44.6 \text{ dB}.$$

Sections 9-7 and 9-8: Radiation by Apertures

Problem 9.23 A uniformly illuminated aperture is of length $l_x = 20\lambda$. Determine the beamwidth between first nulls in the x–z plane.

Solution: The radiation intensity of a uniformly illuminated antenna is given by Eq. (9.90):

$$F(\theta) = \text{sinc}^2(\pi l_x \sin\theta/\lambda) = \text{sinc}^2(\pi\gamma),$$

with

$$\gamma = l_x \sin\theta/\lambda.$$

For $l_x = 20\lambda$,

$$\gamma = 20\sin\theta.$$

The first zero of the sinc function occurs when $\gamma = \pm 1$, as shown in Fig. 9-23. Hence,

$$1 = 20\sin\theta,$$

or

$$\theta = \sin^{-1}\left(\frac{1}{20}\right) = 2.87^\circ,$$

and

$$\beta_{\text{null}} = 2\theta = 5.73^\circ.$$

Problem 9.24 The 10-dB beamwidth is the beam size between the angles at which $F(\theta)$ is 10 dB below its peak value. Determine the 10-dB beamwidth in the x–z plane for a uniformly illuminated aperture with length $l_x = 10\lambda$.

Solution: For a uniformly illuminated antenna of length $l_x = 10\lambda$ Eq. (9.90) gives

$$F(\theta) = \text{sinc}^2(\pi l_x \sin\theta/\lambda) = \text{sinc}^2(10\pi\sin\theta).$$

The peak value of $F(\theta)$ is 1, and the 10-dB level below the peak corresponds to when $F(\theta) = 0.1$ (because $10\log 0.1 = -10$ dB). Hence, we set $F(\theta) = 0.1$ and solve for θ:

$$0.1 = \text{sinc}^2(10\pi\sin\theta).$$

From tabulated values of the sinc function, it follows that the solution of this equation is

$$10\pi \sin\theta = 2.319$$

or

$$\theta \approx 4.23^\circ.$$

Hence, the 10-dB beamwidth is

$$\beta \approx 2\theta = 8.46^\circ.$$

Problem 9.25 A uniformly illuminated rectangular aperture situated in the x–y plane is 2 m high (along x) and 1 m wide (along y). If $f = 10$ GHz, determine

- **(a)** the beamwidths of the radiation pattern in the elevation plane (x–z plane) and the azimuth plane (y–z plane), and
- **(b)** the antenna directivity D in dB.

Solution: From Eqs. (9.94a), (9.94b), and (9.96),

$$\beta_{xz} = 0.88\frac{\lambda}{l_x} = \frac{0.88 \times 3 \times 10^{-2}}{2} = 1.32 \times 10^{-2} \text{ rad} = 0.75^\circ,$$

$$\beta_{yz} = 0.88\frac{\lambda}{l_y} = \frac{0.88 \times 3 \times 10^{-2}}{1} = 2.64 \times 10^{-2} \text{ rad} = 1.51^\circ,$$

$$D = \frac{4\pi}{\beta_{xz}\beta_{yz}} = \frac{4\pi}{(1.32 \times 10^{-2})(2.64 \times 10^{-2})} = 3.61 \times 10^4 = 45.6 \text{ dB}.$$

Problem 9.26 An antenna with a circular aperture has a circular beam with a beamwidth of 3°at 20 GHz.

- **(a)** What is the antenna directivity in dB?
- **(b)** If the antenna area is doubled, what would be the new directivity and new beamwidth?
- **(c)** If the aperture is kept the same as in (a), but the frequency is doubled to 40 GHz, what would the directivity and beamwidth become then?

Solution:

(a) From Eq. (9.96),

$$D \simeq \frac{4\pi}{\beta^2} = \frac{4\pi}{(3^\circ \times \pi/180^\circ)^2} = 4.59 \times 10^3 = 36.6 \text{ dB}.$$

(b) If area is doubled, it means the diameter is increased by $\sqrt{2}$, and therefore the beamwidth decreases by $\sqrt{2}$ to

$$\beta = \frac{3^\circ}{\sqrt{2}} = 2.2^\circ.$$

The directivity increases by a factor of 2, or 3 dB, to $D = 36.6 + 3 = 39.6$ dB.

(c) If f is doubled, λ becomes half as long, which means that the diameter to wavelength ratio is twice as large. Consequently, the beamwidth is half as wide:

$$\beta = \frac{3^\circ}{2} = 1.5^\circ,$$

and D is four times as large, or 6 dB greater, $D = 36.6 + 6 = 42.6$ dB.

Problem 9.27 A 94-GHz automobile collision-avoidance radar uses a rectangular-aperture antenna placed above the car's bumper. If the antenna is 1 m in length and 10 cm in height,

(a) what are its elevation and azimuth beamwidths?

(b) what is the horizontal extent of the beam at a distance of 300 m?

Solution:

(a) At 94 GHz, $\lambda = 3 \times 10^8/(94 \times 10^9) = 3.2$ mm. The elevation beamwidth is $\beta_e = \lambda/0.1\text{ m} = 3.2 \times 10^{-2}$ rad $= 1.8^\circ$. The azimuth beamwidth is $\beta_a = \lambda/1\text{ m} = 3.2 \times 10^{-3}$ rad $= 0.18^\circ$.

(b) At a distance of 300 m, the horizontal extent of the beam is

$$\Delta y = \beta_a R = 3.2 \times 10^{-3} \times 300 = 0.96\text{ m}.$$

Problem 9.28 A microwave telescope consisting of a very sensitive receiver connected to a 100-m parabolic-dish antenna is used to measure the energy radiated by astronomical objects at 20 GHz. If the antenna beam is directed toward the moon and the moon extends over a planar angle of 0.5° from Earth, what fraction of the moon's cross section will be occupied by the beam?

Solution:

$$\beta_{\text{ant}} = \frac{\lambda}{d} = \frac{1.5 \times 10^{-2}}{100} = 1.5 \times 10^{-4}\text{ rad}.$$

For the moon, $\beta_{\text{moon}} = 0.5^\circ \times \pi/180^\circ = 8.73 \times 10^{-3}$ rad. Fraction of the moon's cross section occupied by the beam is

$$\left(\frac{\beta_{\text{ant}}}{\beta_{\text{moon}}}\right)^2 = \left(\frac{1.5 \times 10^{-4}}{8.73 \times 10^{-3}}\right)^2 = 0.3 \times 10^{-3}, \quad \text{or } 0.03\%.$$

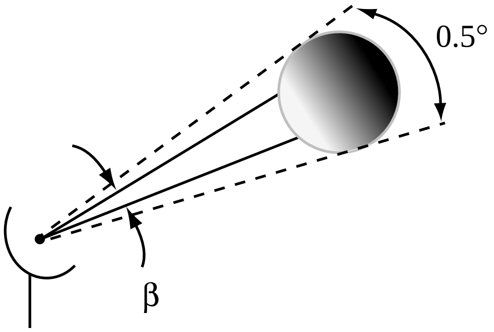


Figure P9.28: Antenna beam viewing the moon.

Sections 9-9 to 9-11: Antenna Arrays

Problem 9.29 A two-element array consisting of two isotropic antennas separated by a distance d along the z-axis is placed in a coordinate system whose z-axis points eastward and whose x-axis points toward the zenith. If a_0 and a_1 are the amplitudes of the excitations of the antennas at $z = 0$ and at $z = d$ respectively, and if δ is the phase of the excitation of the antenna at $z = d$ relative to that of the other antenna, find the array factor and plot the pattern in the x–z plane for

(a) $a_0 = a_1 = 1$, $\delta = \pi/4$, and $d = \lambda/2$,
(b) $a_0 = 1$, $a_1 = 2$, $\delta = 0$, and $d = \lambda$,
(c) $a_0 = a_1 = 1$, $\delta = -\pi/2$, and $d = \lambda/2$,
(d) $a_0 = a_1$, $a_1 = 2$, $\delta = \pi/4$, and $d = \lambda/2$, and
(e) $a_0 = a_1$, $a_1 = 2$, $\delta = \pi/2$, and $d = \lambda/4$.

Solution:

(a) Employing Eq. (9.110),

$$\begin{aligned} F_{\text{a}}(\theta) &= \left| \sum_{i=0}^{1} a_i e^{j\psi_i} e^{jikd\cos\theta} \right|^2 \\ &= |1 + e^{j((2\pi/\lambda)(\lambda/2)\cos\theta + \pi/4)}|^2 \\ &= |1 + e^{j(\pi\cos\theta + \pi/4)}|^2 = 4\cos^2\left(\frac{\pi}{8}(4\cos\theta + 1)\right). \end{aligned}$$

A plot of this array factor pattern is shown in Fig. P9.29(a).

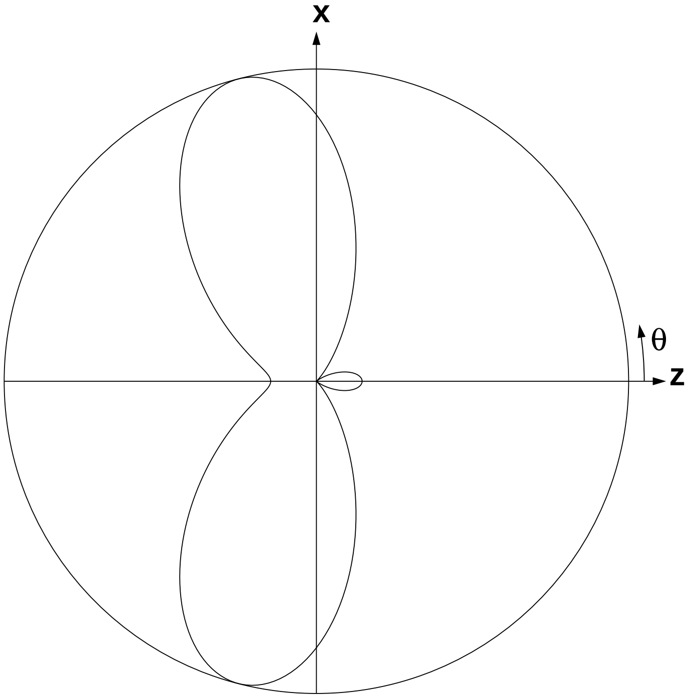


Figure P9.29: (a) Array factor in the elevation plane for Problem 9.29(a).

(b) Employing Eq. (9.110),

$$F_{\mathrm{a}}(\theta) = \left|\sum_{i=0}^{1} a_i e^{j\psi_i} e^{jikd\cos\theta}\right|^2$$
$$= |1+2e^{j((2\pi/\lambda)\lambda\cos\theta+0)}|^2 = |1+2e^{j2\pi\cos\theta}|^2 = 5+4\cos(2\pi\cos\theta).$$

A plot of this array factor pattern is shown in Fig. P9.29(b).

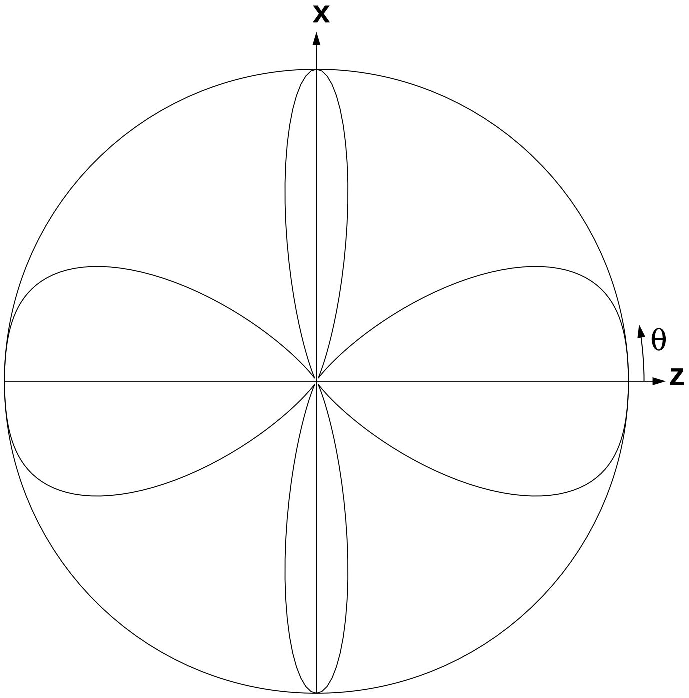


Figure P9.29: (b) Array factor in the elevation plane for Problem 9.29(b).

(c) Employing Eq. (9.110), and setting $a_0 = a_1 = 1$, $\psi = 0$, $\psi_1 = \delta = -\pi/2$ and $d = \lambda/2$, we have

$$\begin{aligned} F_{\rm a}(\theta) &= \left| \sum_{i=0}^{1} a_i e^{j\psi_i} e^{jikd\cos\theta} \right|^2 \\ &= \left| 1 + e^{-j\pi/2} e^{j(2\pi/\lambda)(\lambda/2)\cos\theta} \right|^2 \\ &= \left| 1 + e^{j(\pi\cos\theta - \pi/2)} \right|^2 \\ &= 4\cos^2\left(\frac{\pi}{2}\cos\theta - \frac{\pi}{4}\right). \end{aligned}$$

A plot of the array factor is shown in Fig. P9.29(c).

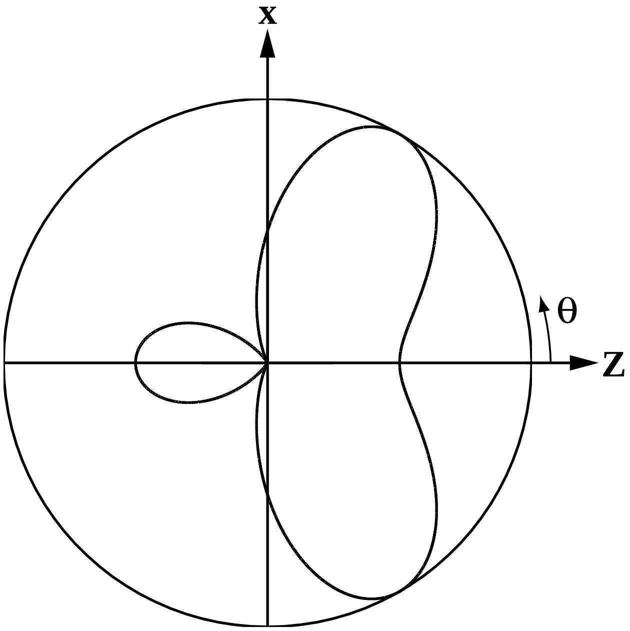


Figure P9.29: (c) Array factor in the elevation plane for Problem 9.29(c).

(d) Employing Eq. (9.110), and setting $a_0 = 1$, $a_1 = 2$, $\psi_0 = 0$, $\psi_1 = \delta = \pi/4$, and $d = \lambda/2$, we have

$$\begin{aligned} F_{\text{a}}(\theta) &= \left| \sum_{i=0}^{1} a_i e^{j\psi_i} e^{jikd\cos\theta} \right|^2 \\ &= \left| 1 + 2e^{j\pi/4} e^{j(2\pi/\lambda)(\lambda/2)\cos\theta} \right|^2 \\ &= \left| 1 + 2e^{j(\pi\cos\theta + \pi/4)} \right|^2 \\ &= 5 + 4\cos\left(\pi\cos\theta + \frac{\pi}{4}\right). \end{aligned}$$

A plot of the array factor is shown in Fig. P9.29(d).

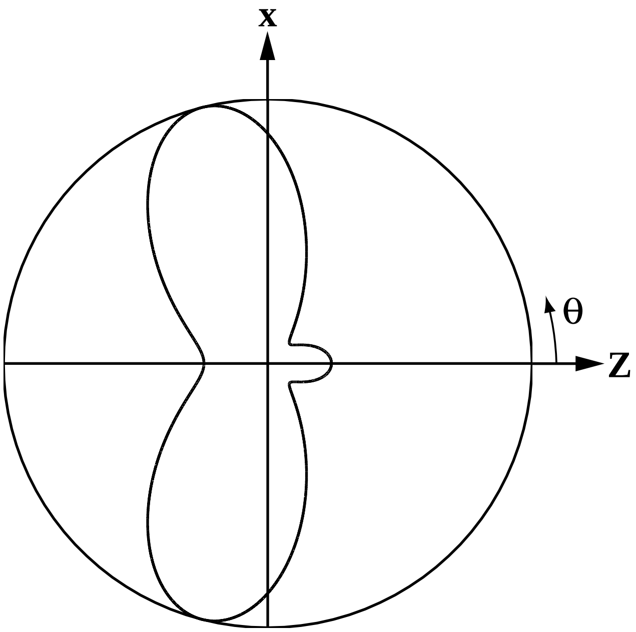


Figure P9.29: (d) Array factor in the elevation plane for Problem 9.29(d).

(e) Employing Eq. (9.110), and setting $a_0 = 1,\ \ a_1 = 2,\ \ \psi_0 = 0,\ \ \psi_1 = \delta = \pi/2$, and $d = \lambda/4$, we have

$$\begin{aligned}
F_{\mathrm{a}}(\theta) &= \left|\sum_{i=0}^{1} a_i e^{j\psi_i} e^{jikd\cos\theta}\right|^2 \\
&= \left|1 + 2e^{j\pi/2} e^{j(2\pi/\lambda)(\lambda/4)\cos\theta}\right|^2 \\
&= \left|1 + 2e^{j(\pi\cos\theta+\pi)/2}\right|^2 \\
&= 5 + 4\cos\left(\frac{\pi}{2}\cos\theta + \frac{\pi}{2}\right) = 5 - 4\sin\left(\frac{\pi}{2}\cos\theta\right).
\end{aligned}$$

A plot of the array factor is shown in Fig. P9.29(e).

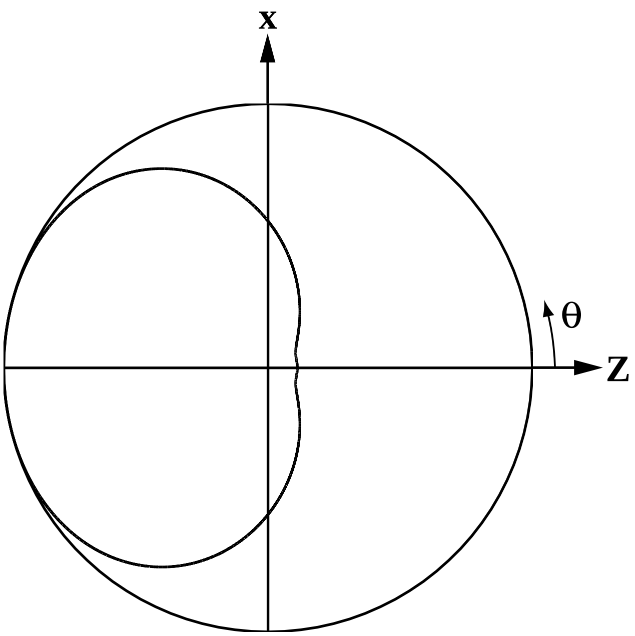


Figure P9.29: (e) Array factor in the elevation plane for Problem 9.29(e).

Problem 9.30 If the antennas in part (a) of Problem 9.29 are parallel vertical Hertzian dipoles with axes along the *x*-direction, determine the normalized radiation intensity in the *x*-*z* plane and plot it.

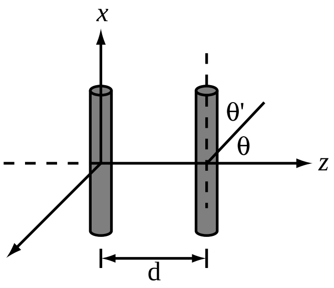


Figure P9.30: (a) Two vertical dipoles of Problem 9.30.

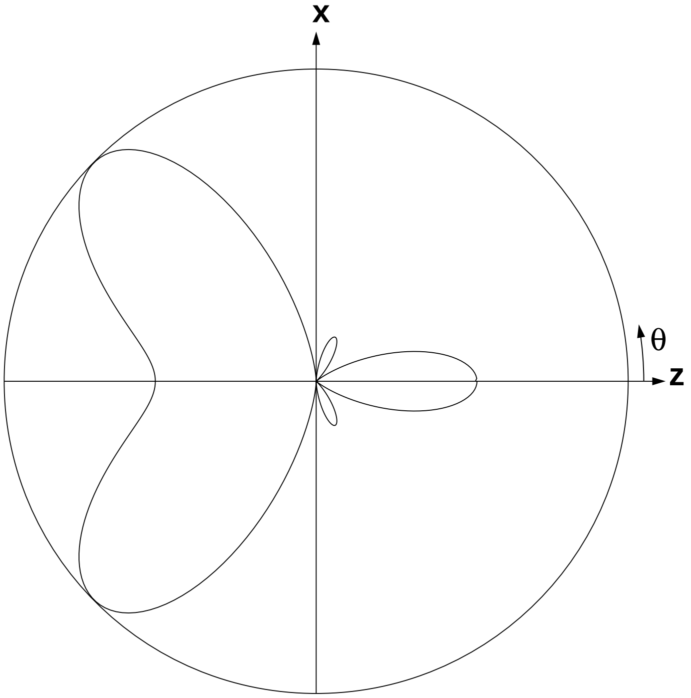


Figure P9.30: (b) Pattern factor in the elevation plane of the array in Problem 9.30(a).

Solution: The power density radiated by a Hertzian dipole is given from Eq. (9.12) by $S_e(\theta') = S_0 \sin^2\theta'$, where θ' is the angle measured from the dipole axis, which in the present case is the *x*-axis (Fig. P9.30).

Hence, $\theta' = \pi/2 - \theta$ and $S_e(\theta) = S_0 \sin^2(\frac{1}{2}\pi - \theta) = S_0 \cos^2\theta$. Then, from Eq. (9.108), the total power density is the product of the element pattern and the array factor. From part (a) of the previous problem:

$$S(\theta) = S_e(\theta) F_a(\theta) = 4S_0 \cos^2\theta \cos^2\left(\frac{\pi}{8}(4\cos\theta + 1)\right).$$

This function has a maximum value of $3.52S_0$ and it occurs at $\theta_{\max} = \pm 135.5°$. The maximum must be found by trial and error. A plot of the normalized array antenna pattern is shown in Fig. P9.30.

Problem 9.31 Consider the two-element dipole array of Fig. 9.29(a). If the two dipoles are excited with identical feeding coefficients ($a_0 = a_1 = 1$ and $\psi_0 = \psi_1 = 0$), choose (d/λ) such that the array factor has a maximum at $\theta = 45°$.

Solution: With $a_0 = a_1 = 1$ and $\psi_0 = \psi_1 = 0$,

$$F_{\text{a}}(\theta) = |1 + e^{j(2\pi d/\lambda)\cos\theta}|^2 = 4\cos^2\left(\frac{\pi d}{\lambda}\cos\theta\right).$$

$F_{\text{a}}(\theta)$ is a maximum when the argument of the cosine function is zero or a multiple of π. Hence, for a maximum at $\theta = 45°$,

$$\frac{\pi d}{\lambda}\cos 45° = n\pi, \qquad n = 0, 1, 2, \ldots .$$

The first value of n, namely $n = 0$, does not provide a useful solution because it requires d to be zero, which means that the two elements are at the same location. While this gives a maximum at $\theta = 45°$, it also gives the same maximum at all angles θ in the y-z plane because the two-element array will have become a single element with an azimuthally symmetric pattern. The value $n = 1$ leads to

$$\frac{d}{\lambda} = \frac{1}{\cos 45°} = 1.414.$$

Problem 9.32 Choose (d/λ) so that the array pattern of the array of Problem 9.31 has a null, rather than a maximum, at $\theta = 45°$.

Solution: With $a_0 = a_1 = 1$ and $\psi_0 = \psi_1 = 0$,

$$F_{\text{a}}(\theta) = |1 + e^{j(2\pi d/\lambda)\cos\theta}|^2 = 4\cos^2\left(\frac{\pi d}{\lambda}\cos\theta\right).$$

$F_{\text{a}}(\theta)$ is equal to zero when the argument of the cosine function is $[(\pi/2) + n\pi]$. Hence, for a null at $\theta = 45°$,

$$\frac{\pi d}{\lambda}\cos 45° = \frac{\pi}{2} + n\pi, \qquad n = 0, 1, 2, \ldots .$$

For $n = 0$,

$$\frac{d}{\lambda} = \frac{1}{2\cos 45°} = 0.707.$$

Problem 9.33 Find and plot the normalized array factor and determine the half-power beamwidth for a five-element linear array excited with equal phase and a uniform amplitude distribution. The interelement spacing is $3\lambda/4$.

Solution: Using Eq. (9.121),

$$F_{\text{an}}(\theta) = \frac{\sin^2[(N\pi d/\lambda)\cos\theta]}{N^2\sin^2[(\pi d/\lambda)\cos\theta]} = \frac{\sin^2[(15\pi/4)\cos\theta]}{25\sin^2[(3\pi/4)\cos\theta]}$$

and this pattern is shown in Fig. P9.33. The peak values of the pattern occur at $\theta = \pm 90°$. From numerical values of the pattern, the angles at which $F_{\text{an}}(\theta) = 0.5$ are approximately 6.75°on either side of the peaks. Hence, $\beta \simeq 13.5°$.

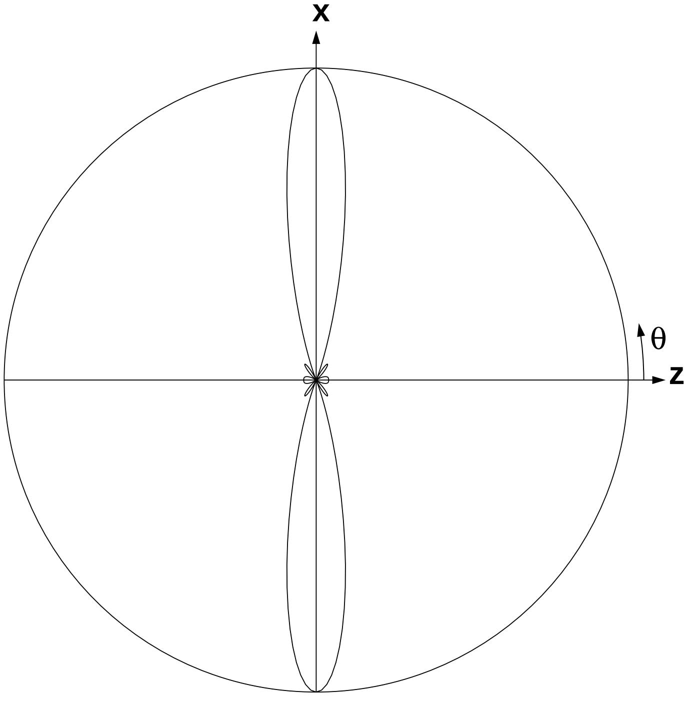


Figure P9.33: Normalized array pattern of a 5-element array with uniform amplitude distribution in Problem 9.33.

Problem 9.34 A three-element linear array of isotropic sources aligned along the z-axis has an interelement spacing of $\lambda/4$ Fig. 9-38 (P9.34). The amplitude excitation of the center element is twice that of the bottom and top elements and the phases are $-\pi/2$ for the bottom element and $\pi/2$ for the top element, relative to that of the center element. Determine the array factor and plot it in the elevation plane.

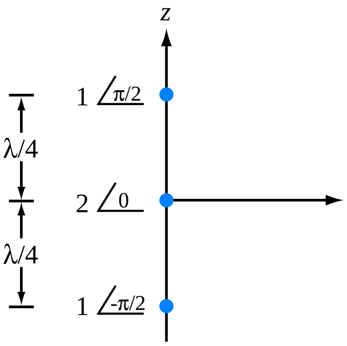


Figure P9.34: (a) Three-element array of Problem 9.34.

Solution: From Eq. (9.110),

$$\begin{aligned}
F_{\rm a}(\theta) &= \left|\sum_{i=0}^{2} a_i e^{j\psi_i} e^{jikd\cos\theta}\right|^2 \\
&= \left|a_0 e^{j\psi_0} + a_1 e^{j\psi_1} e^{jkd\cos\theta} + a_2 e^{j\psi_2} e^{j2kd\cos\theta}\right|^2 \\
&= \left|e^{j(\psi_1-\pi/2)} + 2e^{j\psi_1} e^{j(2\pi/\lambda)(\lambda/4)\cos\theta} + e^{j(\psi_1+\pi/2)} e^{j2(2\pi/\lambda)(\lambda/4)\cos\theta}\right|^2 \\
&= \left|e^{j\psi_1} e^{j(\pi/2)\cos\theta}\right|^2 \left|e^{-j\pi/2} e^{-j(\pi/2)\cos\theta} + 2 + e^{j\pi/2} e^{j(\pi/2)\cos\theta}\right|^2 \\
&= 4\left(1+\cos\left(\tfrac{1}{2}\pi(1+\cos\theta)\right)\right)^2, \\
F_{\rm an}(\theta) &= \tfrac{1}{4}\left(1+\cos\left(\tfrac{1}{2}\pi(1+\cos\theta)\right)\right)^2.
\end{aligned}$$

This normalized array factor is shown in Fig. P9.34.

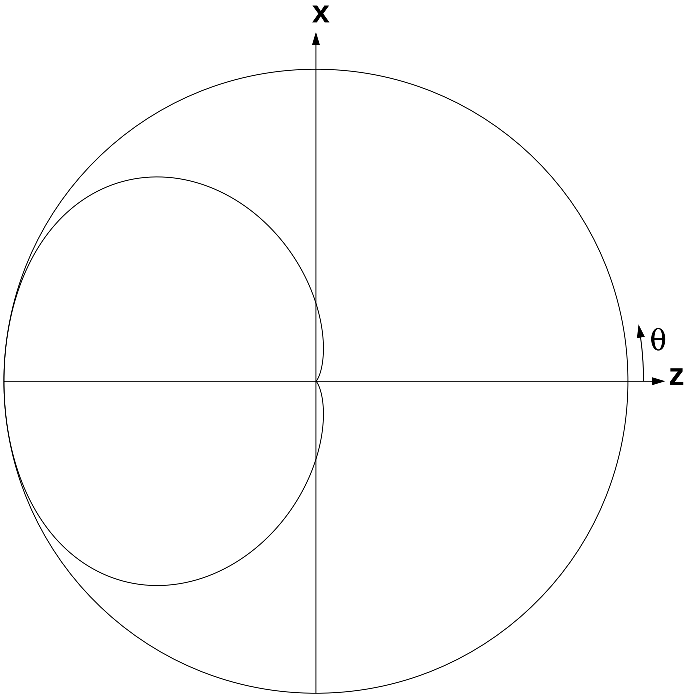


Figure P9.34: (b) Normalized array pattern of the 3-element array of Problem 9.34.

Problem 9.35 An eight-element linear array with $\lambda/2$ spacing is excited with equal amplitudes. To steer the main beam to a direction 60° below the broadside direction, what should be the incremental phase delay between adjacent elements? Also, give the expression for the array factor and plot the pattern.

Solution: Since broadside corresponds to $\theta = 90°$, 60° below broadside is $\theta_0 = 150°$. From Eq. (9.125),

$$\delta = kd\cos\theta_0 = \frac{2\pi}{\lambda}\frac{\lambda}{2}\cos 150° = -2.72\ (\text{rad}) = -155.9°.$$

Combining Eq. (9.126) with (9.127) gives

$$F_{\text{an}}(\theta) = \frac{\sin^2\left(\frac{1}{2}Nkd(\cos\theta - \cos\theta_0)\right)}{N^2\sin^2\left(\frac{1}{2}kd(\cos\theta - \cos\theta_0)\right)} = \frac{\sin^2\left(4\pi(\cos\theta + \frac{1}{2}\sqrt{3})\right)}{64\sin^2\left(\frac{1}{2}\pi(\cos\theta + \frac{1}{2}\sqrt{3})\right)}.$$

The pattern is shown in Fig. P9.35.

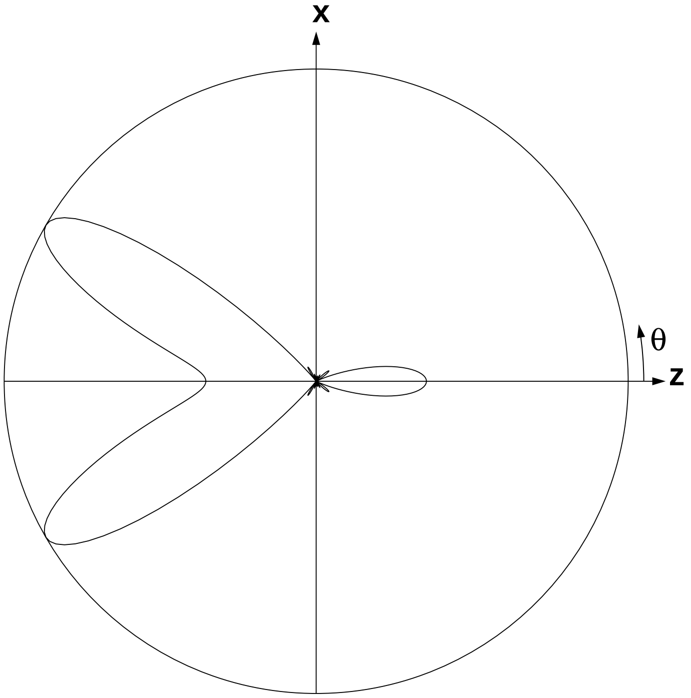


Figure P9.35: Pattern of the array of Problem 9.35.

Problem 9.36 A linear array arranged along the z-axis consists of 12 equally spaced elements with $d = \lambda/2$. Choose an appropriate incremental phase delay δ so as to steer the main beam to a direction 30° above the broadside direction. Provide an expression for the array factor of the steered antenna and plot the pattern. From the pattern, estimate the beamwidth.

Solution: Since broadside corresponds to $\theta = 90^\circ$, 30° above broadside is $\theta_0 = 60^\circ$. From Eq. (9.125),

$$\delta = kd\cos\theta_0 = \frac{2\pi}{\lambda}\frac{\lambda}{2}\cos 60^\circ = 1.57\ (\text{rad}) = 90^\circ.$$

Combining Eq. (9.126) with (9.127) gives

$$F_{\text{an}}(\theta) = \frac{\sin^2\left(\frac{1}{2}12kd(\cos\theta - \cos\theta_0)\right)}{144\sin^2\left(\frac{1}{2}kd(\cos\theta - \cos\theta_0)\right)} = \frac{\sin^2\left(6\pi(\cos\theta - 0.5)\right)}{144\sin^2\left(\frac{\pi}{2}(\cos\theta - 0.5)\right)}\,.$$

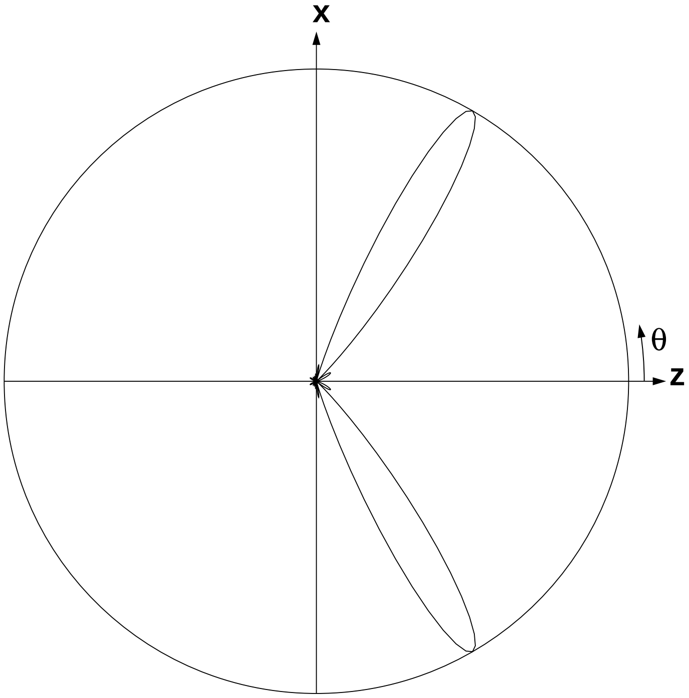


Figure P9.36: Array pattern of Problem 9.36.

The pattern is shown in Fig. P9.36. The beamwidth is $\approx 10^\circ$.

Problem 9.37 A 50-cm long dipole is excited by a sinusoidally varying current with an amplitude $I_0 = 5$ A. Determine the time average power radiated by the dipole if the oscillating frequency is:

(a) 1 MHz,
(b) 300 MHz.

Solution:

(a) At 1 MHz,

$$\lambda = \frac{3 \times 10^8}{10^6} = 300 \text{ m}.$$

Hence, the dipole length satisfies the “short” dipole criterion ($l \leq \lambda/50$).

Using (9.34),

$$P_{\text{rad}} = 40\pi^2 I_0^2 \left(\frac{l}{\lambda}\right)^2$$
$$= 40\pi^2 \times 5^2 \times \left(\frac{0.5}{300}\right)^2 = 27.4 \text{ mW}.$$

(b) At 300 MHz,

$$\lambda = \frac{3 \times 10^8}{3 \times 10^8} = 1 \text{ m}.$$

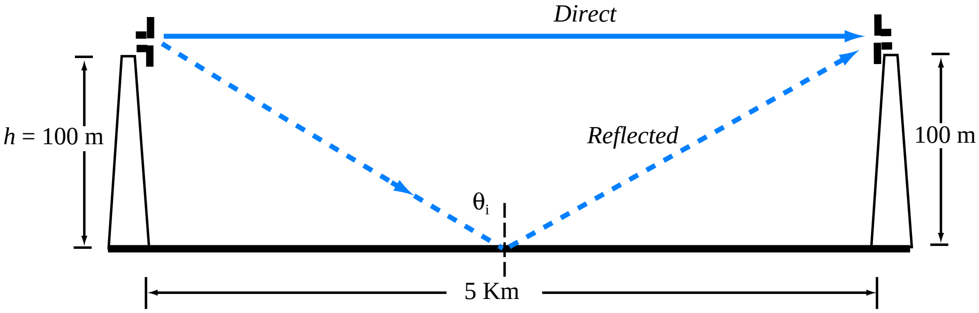
Hence, the dipole is $\lambda/2$ in length, in which case we can use (9.46) to calculate P_{rad}:

$$P_{\text{rad}} = 36.6 I_0^2 = 36.6 \times 5^2 = 915 \text{ W}.$$

Thus, at the higher frequency, the antenna radiates $[915/27.3 \times 10^{-3})] = 33{,}516.5$ times as much power as it does at the lower frequency!

Problem 9.38 The configuration shown in the figure depicts two vertically oriented half-wave dipole antennas pointed towards each other, with both positioned on 100-m-tall towers separated by a distance of 5 km. If the transit antenna is driven by a 50-MHz current with amplitude $I_0 = 2$ A, determine:

- **(a)** The power received by the receive antenna in the absence of the surface. (Assume both antennas to be lossless.)
- **(b)** The power received by the receive antenna after incorporating reflection by the ground surface, assuming the surface to be flat and to have $\varepsilon_r = 9$ and conductivity $\sigma = 10^{-3}$ (S/m).



Solution:

(a) Since both antennas are lossless,

$$P_{\text{rec}} = P_{\text{int}} = S_i A_{\text{er}}$$

where S_i is the incident power density and A_{er} is the effective area of the receive dipole. From Section 9-3,

$$S_i = S_0 = \frac{15I_0^2}{\pi R^2},$$

and from (9.64) and (9.47),

$$A_{er} = \frac{\lambda^2 D}{4\pi} = \frac{\lambda^2}{4\pi} \times 1.64 = \frac{1.64\lambda^2}{4\pi}.$$

Hence,

$$P_{rec} = \frac{15I_0^2}{\pi R^2} \times \frac{1.64\lambda^2}{4\pi} = 3.6 \times 10^{-6} \text{ W}.$$

(b) The electric field of the signal intercepted by the receive antenna now consists of a direct component, E_d, due to the directly transmitted signal, and a reflected component, E_r, due to the ground reflection. Since the power density S and the electric field E are related by

$$S = \frac{|E|^2}{2\eta_0},$$

it follows that

$$\begin{aligned} E_d &= \sqrt{2\eta_0 S_i}\, e^{-jkR} \\ &= \sqrt{2\eta_0 \times \frac{15I_0^2}{\pi R^2}}\, e^{-jkR} \\ &= \sqrt{\frac{30\eta_0}{\pi}}\, \frac{I_0}{R}\, e^{-jkR} \end{aligned}$$

where the phase of the signal is measured with respect to the location of the transmit antenna, and $k = 2\pi/\lambda$. Hence,

$$E_d = 0.024e^{-j120^\circ} \quad \text{(V/m)}.$$

The electric field of the reflected signal is similar in form except for the fact that R should be replaced with R', where R' is the path length traveled by the reflected signal, and the electric field is modified by the reflection coefficient Γ. Thus,

$$E_r = \left(\sqrt{\frac{30\eta_0}{\pi}}\, \frac{I_0}{R'}\, e^{-jkR'}\right)\Gamma.$$

From the problem geometry

$$R' = 2\sqrt{(2.5 \times 10^3)^2 + (100)^2} = 5004.0 \text{ m}.$$

Since the dipole is vertically oriented, the electric field is parallel polarized. To calculate Γ, we first determine

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r} = \frac{10^{-3}}{2\pi \times 50 \times 10^6 \times 8.85 \times 10^{-12} \times 9} = 0.04.$$

From Table 7-1,

$$\eta_c \approx \eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{\eta_0}{\sqrt{9}} = \frac{\eta_0}{3} .$$

From (8.66a),

$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

From the geometry,

$$\begin{aligned}
\cos\theta_i &= \frac{h}{(R'/2)} = \frac{100}{2502} = 0.04 \\
\theta_i &= 87.71^\circ \\
\theta_t &= \sin^{-1}\left(\frac{\sin\theta_i}{\sqrt{\varepsilon_r}}\right) = 19.46^\circ \\
\eta_1 &= \eta_0 \text{ (air)} \\
\eta_2 &= \eta = \frac{\eta_0}{3} .
\end{aligned}$$

Hence,

$$\Gamma_\parallel = \frac{(\eta_0/3) \times 0.94 - \eta_0 \times 0.04}{(\eta_0/3) \times 0.94 + \eta_0 \times 0.04} = 0.77.$$

The reflected electric field is

$$\begin{aligned}
E_r &= \left(\sqrt{\frac{30\eta_0}{\pi}}\, \frac{I_0}{R'}\, e^{-jkR'}\right)\Gamma \\
&= 0.018e^{j0.6^\circ} \quad \text{(V/m)}.
\end{aligned}$$

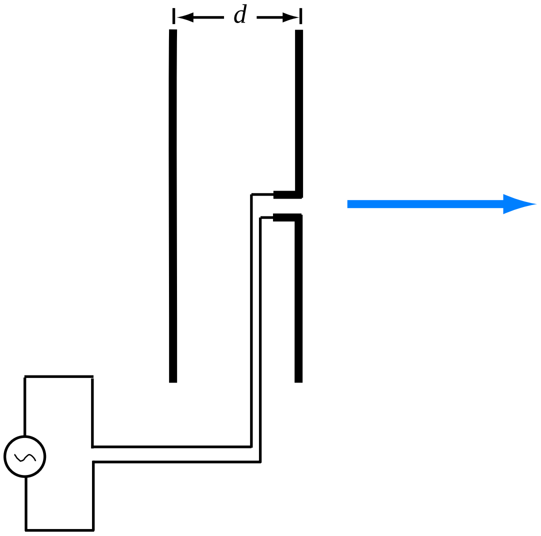
The total electric field is

$$\begin{aligned}
E &= E_d + E_r \\
&= 0.024e^{-j120^\circ} + 0.018e^{j0.6^\circ} \\
&= 0.02e^{-j73.3^\circ} \quad \text{(V/m)}.
\end{aligned}$$

The received power is

$$
\begin{aligned}
P_{\text{rec}} &= S_i A_{\text{er}} \\
&= \frac{|E|^2}{2\eta_0} \times \frac{1.64\lambda^2}{4\pi} \\
&= 2.5 \times 10^{-6} \text{ W}.
\end{aligned}
$$

Problem 9.39

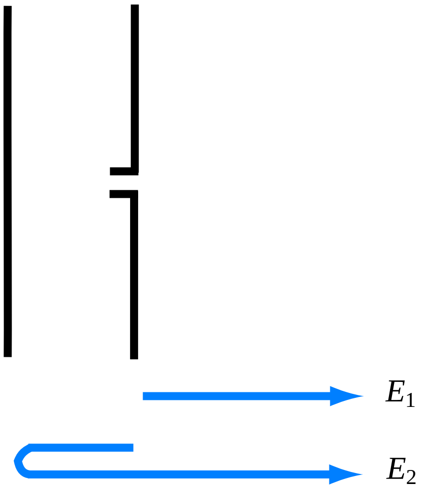


The figure depicts a half-wave dipole connected to a generator through a matched transmission line. The directivity of the dipole can be modified by placing a reflecting rod a distance *d* behind the dipole. What would its reflectivity in the forward direction be if:

(a) $d = \lambda/4$,

(b) $d = \lambda/2$.

Solution: Without the reflecting rod, the directivity of a half-wave dipole is 1.64 (see 9.47). When the rod is present, the wave moving in the direction of the arrow consists of two electric field components:



$$E = E_1 + E_2,$$

where E_1 is the field of the radiated wave moving to the right and E_2 is the field that initally moved to the left and then got reflected by the rod. The two are essentially equal in magnitude, but E_2 lags in phase by $2kd$ relative to E_1, and also by π because the reflection coefficient of the metal rod is -1. Hence, we can write E at any point to the right of the antenna as

$$\begin{aligned} E &= E_1 + E_1 e^{j\pi} e^{-j2kd} \\ &= E_1(1 + e^{-j(2kd-\pi)}) \end{aligned}$$

(a) For $d = \lambda/4,\ \ 2kd = 2 \cdot \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \pi$.

$$E = E_1(1 + e^{-j(\pi-\pi)}) = 2E_1.$$

The directivity is proportional to power, or $|E|^2$. Hence, D will increase by a factor of 4 to

$$D = 1.64 \times 4 = 6.56.$$

(b) For $d = \lambda/2,\ \ 2kd = 2\pi$.

$$E = E_1(1-1) = 0.$$

Thus, the antenna radiation pattern will have a null in the forward direction.

Problem 9.40 A five-element equally spaced linear array with $d = \lambda/2$ is excited with uniform phase and an amplitude distribution given by the binomial distribution

$$a_i = \frac{(N-1)!}{i!(N-i-1)!}\,, \qquad i = 0, 1, \ldots, N-1$$

where N is the number of elements. Develop an expression for the array factor.

Solution: Using the given formula,

$$\begin{aligned}
a_0 &= \frac{(5-1)!}{0!4!} = 1 \qquad \text{(note that } 0! = 1\text{)}\\
a_1 &= \frac{4!}{1!3!} = 4\\
a_2 &= \frac{4!}{2!2!} = 6\\
a_3 &= \frac{4!}{3!1!} = 4\\
a_4 &= \frac{4!}{0!4!} = 1
\end{aligned}$$

Application of (9.113) leads to:

$$\begin{aligned}
F_a(\gamma) &= \left|\sum_{i=0}^{N-1} a_i e^{ji\gamma}\right|^2, \qquad \gamma = \frac{2\pi d}{\lambda}\cos\theta\\
&= \left|1 + 4e^{j\gamma} + 6e^{j2\gamma} + 4e^{j3\gamma} + e^{j4\gamma}\right|^2\\
&= \left|e^{j2\gamma}(e^{-j2\gamma} + 4e^{-j\gamma} + 6 + 4e^{j\gamma} + e^{j2\gamma})\right|^2\\
&= (6 + 8\cos\gamma + 2\cos 2\gamma)^2.
\end{aligned}$$

With $d = \lambda/2$, $\gamma = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}\cos\theta = \pi\cos\theta$,

$$F_a(\theta) = [6 + 8\cos(\pi\cos\theta) + 2\cos(2\pi\cos\theta)]^2.$$

Chapter 10: Satellite Communication Systems and Radar Sensors

Lesson #70 and 71

Chapter — Section: 10-1 to 10-4

Topics: Communication systems

Highlights:

- Geosynchronous orbit
- Transponders, frequency allocations
- Power budgets
- Antennas

Lesson #72 and 73

Chapter — Section: 10-5 to 10-8

Topics: Radar systems

Highlights:

- Acronym for RADAR
- Range and azimuth resolutions
- Detection of signal against noise
- Doppler
- Monopulse radar

Chapter 10

Sections 10-1 to 10-4: Satellite Communication Systems

Problem 10.1 A remote sensing satellite is in circular orbit around the earth at an altitude of 1,100 km above the earth's surface. What is its orbital period?

Solution: The orbit's radius is $R_0 = R_\mathrm{e} + h = 6,378 + 1,100 = 7478$ km. Rewriting Eq. (10.6) for T:

$$T = \left(\frac{4\pi^2 R_0^3}{GMe}\right)^{1/2} = \left[\frac{4\pi^2 \times (7.478\times10^6)^3}{6.67\times10^{-11}\times5.98\times10^{24}}\right]^{1/2}$$
$$= 4978.45 \text{ s} = 82.97 \text{ minutes.}$$

Problem 10.2 A transponder with a bandwidth of 400 MHz uses polarization diversity. If the bandwidth allocated to transmit a single telephone channel is 4 kHz, how many telephone channels can be carried by the transponder?

Solution: Number of telephone channels $= \dfrac{2\times 400 \text{ MHz}}{4 \text{ kHz}} = \dfrac{2\times4\times10^8}{4\times10^3} = 2\times10^5$ channels.

Problem 10.3 Repeat Problem 10.2 for TV channels, each requiring a bandwidth of 6 MHz.

Solution: Number of telephone channels $= \dfrac{2\times4\times10^8}{6\times10^6} = 133.3 \simeq 133$ channels. We need to round down becasue we cannot have a partial channel.

Problem 10.4 A geostationary satellite is at a distance of 40,000 km from a ground receiving station. The satellite transmitting antenna is a circular aperture with a 1-m diameter and the ground station uses a parabolic dish antenna with an effective diameter of 20 cm. If the satellite transmits 1 kW of power at 12 GHz and the ground receiver is characterized by a system noise temperature of 1,000 K, what would be the signal-to-noise ratio of a received TV signal with a bandwidth of 6 MHz? The antennas and the atmosphere may be assumed lossless.

Solution: We are given

$$R = 4\times10^7 \text{ m},\qquad d_\mathrm{t} = 1 \text{ m},\qquad d_\mathrm{r} = 0.2 \text{ m},\qquad P_\mathrm{t} = 10^3 \text{ W},$$
$$f = 12 \text{ GHz},\qquad T_\mathrm{sys} = 1,000 \text{ K},\qquad B = 6 \text{ MHz}.$$

At $f = 12$ GHz, $\lambda = c/f = 3\times 10^8/12\times 10^9 = 2.5\times 10^{-2}$ m. With $\xi_t = \xi_r = 1$,

$$G_t = D_t = \frac{4\pi A_t}{\lambda^2} = \frac{4\pi(\pi d_t^2/4)}{\lambda^2} = \frac{4\pi\times\pi\times 1}{4\times(2.5\times 10^{-2})^2} = 15,791.37,$$
$$G_r = D_r = \frac{4\pi A_r}{\lambda^2} = \frac{4\pi(\pi d_r^2/4)}{\lambda^2} = \frac{4\pi\times\pi(0.2)^2}{4\times(2.5\times 10^{-2})^2} = 631.65.$$

Applying Eq. (10.11) with $\Upsilon(\theta) = 1$ gives:

$$S_n = \frac{P_t G_t G_r}{K T_{sys} B}\left(\frac{\lambda}{4\pi R}\right)^2 = \frac{10^3\times 15,791.37\times 631.65}{1.38\times 10^{-23}\times 10^3\times 6\times 10^6}\left(\frac{2.5\times 10^{-2}}{4\pi\times 4\times 10^7}\right)^2 = 298.$$

Sections 10-5 to 10-8: Radar Sensors

Problem 10.5 A collision avoidance automotive radar is designed to detect the presence of vehicles up to a range of 0.5 km. What is the maximum usable PRF?

Solution: From Eq. (10.14),

$$f_p = \frac{c}{2R_u} = \frac{3\times 10^8}{2\times 0.5\times 10^3} = 3\times 10^5 \text{ Hz}.$$

Problem 10.6 A 10-GHz weather radar uses a 15-cm-diameter lossless antenna. At a distance of 1 km, what are the dimensions of the volume resolvable by the radar if the pulse length is 1 μs?

Solution: Resolvable volume has dimensions $\Delta x, \Delta y$, and ΔR.

$$\Delta x = \Delta y = \beta R = \frac{\lambda}{d}R = \frac{3\times 10^{-2}}{0.15}\times 10^3 = 200 \text{ m},$$
$$\Delta R = \frac{c\tau}{2} = \frac{3\times 10^8}{2}\times 10^{-6} = 150 \text{ m}.$$

Problem 10.7 A radar system is characterized by the following parameters: $P_t = 1$ kW, $\tau = 0.1$ μs, $G = 30$ dB, $\lambda = 3$ cm, and $T_{sys} = 1,500$ K. The radar cross section of a car is typically 5 m^2. How far can the car be and remain detectable by the radar with a minimum signal-to-noise ratio of 13 dB?

Solution: $S_{\text{min}} = 13$ dB means $S_{\text{min}} = 20$. $G = 30$ dB means $G = 1000$. Hence, by Eq. (10.27),

$$R_{\text{max}} = \left[\frac{P_{\text{t}}\tau G^2\lambda^2\sigma_{\text{t}}}{(4\pi)^3 K T_{\text{sys}} S_{\text{min}}}\right]^{1/4}$$
$$= \left[\frac{10^3 \times 10^{-7} \times 10^6 \times (3\times 10^{-2})^2 \times 5}{(4\pi)^3 \times 1.38 \times 10^{-23} \times 1.5 \times 10^3 \times 20}\right]^{1/4} = 4837.8 \text{ m} = 4.84 \text{ km}.$$

Problem 10.8 A 3-cm-wavelength radar is located at the origin of an x–y coordinate system. A car located at $x = 100$ m and $y = 200$ m is heading east (x-direction) at a speed of 120 km/hr. What is the Doppler frequency measured by the radar?

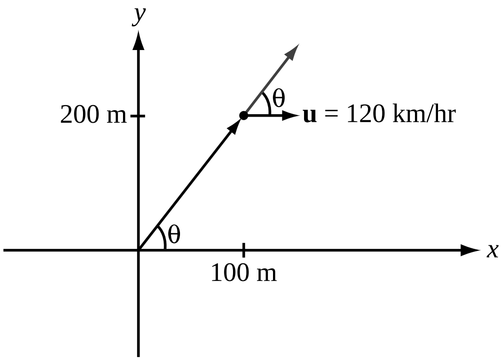


Figure P10.8: Geometry of Problem 10.8.

Solution:

$$\theta = \tan^{-1}\left(\frac{200}{100}\right) = 63.43^\circ,$$
$$u = 120 \text{ km/hr} = \frac{1.2\times 10^5}{3600} = 33.33 \text{ m/s},$$
$$f_{\text{d}} = \frac{-2u}{\lambda}\cos\theta = \frac{-2\times 33.33}{3\times 10^{-2}}\cos 63.43^\circ = -993.88 \text{ Hz}.$$